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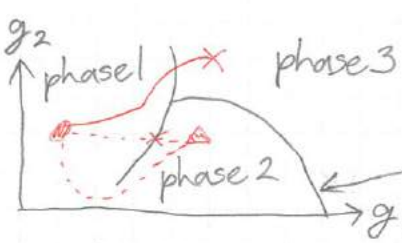
10/16 Generalized Symmetries and Application to Gauge Theories

(In Nov. Ohmori-san will give a lecture on non-inv. sym.)

QFT ← universal effective descriptions for quantum many-body systems.
But, they are often strongly-coupled.

⇒ We'd like to extract a good data characterizing quantum phases of matte



space of QFTs
= space of coupling constant

phase boundaries = locations of quantum phase transitions

Def. phase 1 and 3 are in different quantum phases if we encounter quantum phase transitions at least once for any continuous paths connecting these two points.

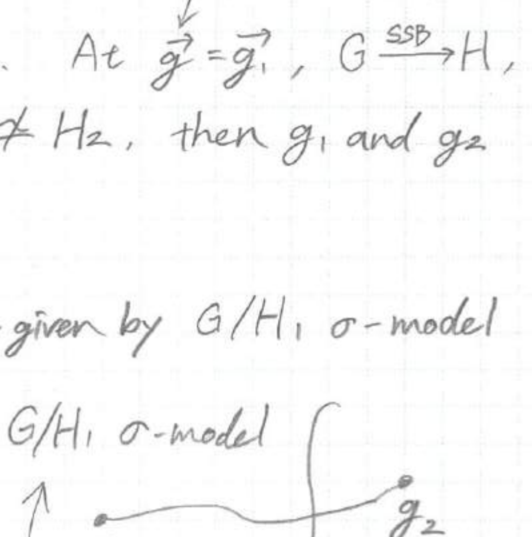
Although quantum phases of matters are defined, we need "some invariants" to classify quantum phases in practice.
↳ symmetries are playing important roles.

cf.) Landau criterion

Landau criterion

Assume the system has a global sym. G. At $\vec{g} = \vec{g_1}$ (p.t. of space of coupling), $G \xrightarrow{SSB} H_1$, and at $\vec{g} = \vec{g_2}$, $G \xrightarrow{SSB} H_2$. If $H_1 \not\simeq H_2$, then g_1 and g_2 belong to different phases.

Idea. Low energy theories at g_1 and g_2 are given by G/H_1 σ-model and G/H_2 σ-model, resp.
Since there's no continuous space of effective couplings connecting these two descriptions then has to be a phase transition.



There's no continuous parameters connecting these two.

Def. of SSB

① Add sym-breaking perturbation $\mathcal{L} + \varepsilon \mathcal{O}$, take $V \to \infty$, and then $\varepsilon \to 0$.

$$\langle \mathcal{O} \rangle := \lim_{\varepsilon \to 0} \lim_{V \to \infty} \langle \mathcal{O} \rangle_V = v \neq 0 \Rightarrow \text{Sym. is broken.}$$

② ODLRO (off-diagonal long range order)

$$\lim_{\substack{V \to \infty \\ |x-y| \to \infty}} \langle \mathcal{O}(x) \mathcal{O}^*(y) \rangle_{\text{connected}} = v^2 \neq 0 \rightsquigarrow \text{sym. is broken}$$

(Violation of cluster decomposition)

$$\text{cf.)} \quad \langle \overset{\swarrow \mathcal{O} \text{ is nontrivial rep. of } G}{\mathcal{O}} \rangle_V = 0$$

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle_{\text{conn.}} = \langle \mathcal{O}(x) \overset{\swarrow v^2}{\mathcal{O}}(y) \rangle - \langle \overset{\swarrow 0}{\mathcal{O}(x)} \rangle \langle \mathcal{O}(y) \rangle$$
$$= \langle \mathcal{O}(x) \mathcal{O}(y) \rangle$$

Ising case

$$H = -J \sum_{\langle x, x' \rangle} S(x) S(x')$$

$$|+\rangle = |+++\cdots\rangle$$

$$|-\rangle = |---\cdots\rangle$$

$$\rightsquigarrow |GS\rangle_V = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$$

$$\langle S(x) \rangle_V = \frac{1}{2} \left(\langle + | S | + \rangle + \langle - | S | - \rangle \right) = 0$$

Motivation for generalized sym.

↳ We'd like to apply this idea (like Landau criterion) for gauge theories

However, gauge theories does not necessarily have a good sym. to characterize its phases.

↳ There's actually a good generalization of syms. providing new invariants.

Definition of generalized symmetries

Def (global) Generalized sym. in QFT

↑ Euclidean
local
relativistic

= Data of topological operators in QFT.

↑

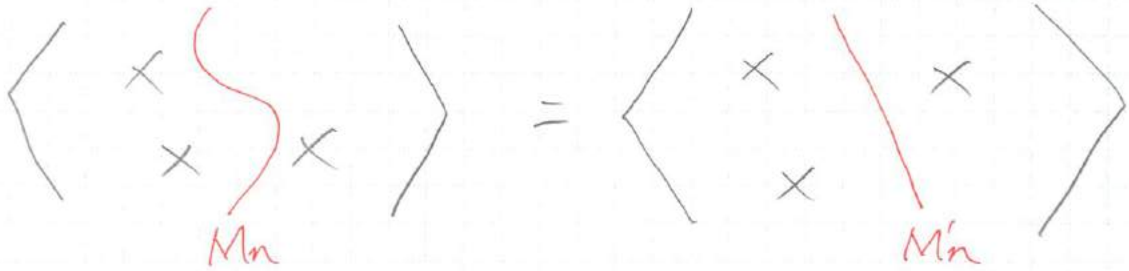
topological op. = extended op. $\mathcal{O}(M_n)$

$\uparrow$ n-dim sub mfd of spacetime

s.t.

$$\langle \mathcal{O}(M_n) \cdots \rangle \underset{\uparrow}{=} \langle \mathcal{O}(M'_n) \cdots \rangle$$

if M_n and M'_n are conti. deformable from one to another.



Ex. 1 (Continuous global sym.)

d-dim QFT has a conti. global sym. G.

⇒ We have a Noether current $j_a^\mu(x)$

s.t. $\partial_\mu j_a^\mu(x) = 0$ as operator identity.

We can construct a topological op. (= Noether charge)

$$U_\varepsilon(\underset{\substack{\uparrow \\ \text{codim-1 mfd}}}{M_{d-1}}) = \exp\Big(i\varepsilon^a \int_{M_{d-1}} \underset{\substack{\uparrow \\ d-1 \text{ form}}}{j_a}\Big)$$

(※通常の$*j$をjと書いている)

$$\frac{\partial}{\partial \tau} Q_a(\tau) = \int d^{d-1}x \, \underbrace{\partial_0 j^0}_{\overset{\shortparallel}{\nabla \cdot \vec{j}}}$$

↙

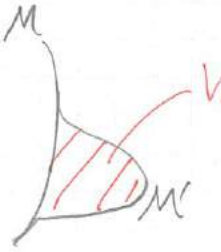
Comment 1. This is a geralization of usual Noether charge $Q_a(\tau) = \int_{\mathbb{R}^{d-1}} d^{d-1}x \, j_a^0(\tau, \vec{x})$

If we take $M_{d-1} = \{\tau\} \times \mathbb{R}^{d-1}$, $U_\varepsilon(\{\tau\} \times \mathbb{R}^{d-1}) = e^{i\varepsilon_a Q_a}$

comment 2. $U_\varepsilon(M_{d-1})$ is topological.

$$M_{d-1} \xrightarrow{\text{conti. deform}} M'_{d-1}$$

$$\int_M j - \int_{M'} j = \int_{\partial V} j \overset{\text{stokes}}{=} \int_V \underbrace{dj}_{\frac{1}{d!}\partial_\mu j^\mu d^d x} \overset{\text{as op. identity}}{=} 0$$



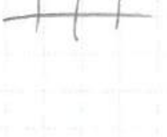
V_d : d-dim volume wiped through conti. deformation

Comment 3. Topologicalness of $U_\varepsilon(M_{d-1})$ of any M_{d-1} is equivalent to the local conservation $\partial_\mu j^\mu = 0$

Ex. 2 (Discrete sym.)

Here let's take lattice description, instead of continuous fields.
In particular, we take an example of complex scalar fields.

$$S[\vec{\varphi}] = \sum_{\langle x, x' \rangle} J\, \vec{\varphi}^*_{(x)} \cdot \vec{\varphi}(x') + \sum_x V(\vec{\varphi}_{(x)}, \vec{\varphi}^*_{(x)})$$



Assume that this has a sym. $G \leftarrow \begin{cases} \text{discrete} \\ \text{continuous} \end{cases}$

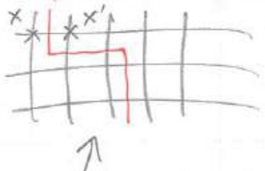
$\left(\Leftrightarrow g \in G \quad , \quad \vec{\varphi} \to g \cdot \vec{\varphi} \right)$

↑ realized as $U(N)$ matrix and $V(g\cdot\varphi_{(x)}, \varphi^\dagger_{(x)} g^\dagger) = V(\varphi_{(x)}, \varphi^\dagger_{(x)})$

Now, we construct a codim-1 topological op. corresponding to $g \in G$

(Write it in eq. is but idea is simple. ← We just draw figures.)

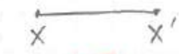
← take a (d-1) dim surface on a dual lattice



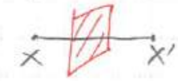
↑ surface is taken that it crosses bonds instead of sites.

We define bond-dependent hopping

$J\langle x, x' \rangle = \begin{cases} J & \text{if} \\ J \times g & \text{if} \end{cases}$



not crossed by surface



$S[\varphi, \varphi'; U_g($▨$)]$

$$= \sum_{\bullet\!-\!\bullet} J\,\varphi^\dagger_{(x)} \cdot \varphi_{(x')} + \sum_{\bullet\!-\!\text{▨}\!-\!\bullet} J\,\varphi^\dagger_{(x)} \cdot g \cdot \varphi_{(x')}$$

$$+ \sum_{x} V(\varphi_{(x)}, \varphi^\dagger_{(x)})$$

We define $U_g(M_{d-1})$ as

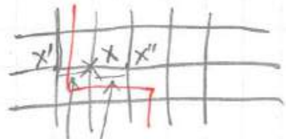
$$\langle U_g(M_{d-1})\, \mathcal{O}_{x_1} \cdots \mathcal{O}_{x_n} \rangle = \int \mathcal{D}\varphi\, e^{-S[\varphi, \varphi'; U_g(M_{d-1})]} \cdot \mathcal{O}_{x_1} \cdots \mathcal{O}_{x_n}$$

(This type of def. of ops are often called "defect" op.s)

Theorem

This defect op. $U_g(M_{d-1})$ is topological

(Proof)

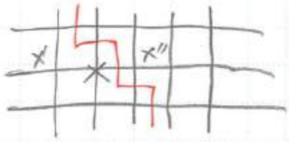


for this $\varphi(x)$, ← This is just a dummy variable of path int.

perform the change of variable

$$\varphi(x) \to g^\dagger \cdot \varphi(x)$$

$$J\varphi^\dagger_{(x')}\, g \cdot \varphi(x) \longrightarrow J\varphi^\dagger_{(x')} \cdot \varphi(x)$$

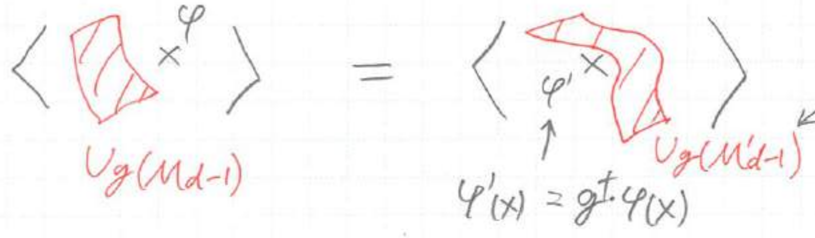


$$J\varphi^\dagger(x) \cdot \varphi(x'') \longrightarrow J(g^\dagger \varphi(x))^\dagger \cdot \varphi(x'')$$

$$= J\varphi^\dagger(x) \cdot g \cdot \varphi(x'')$$

$$\Rightarrow \langle U_g(M_{d-1})\, \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle U_g(M'_{d-1})\, \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$$

$\to U_g(M_{d-1})$ is topological



This is an analogue of Noether charge $Q(M_{d-1})$

$U_{g=e^{i\epsilon}} = e^{i\epsilon Q}$

$[\varphi(x)\, Q] = q \cdot \varphi(x)$

2コマ目 From these examples (Ex 1 and 2), we've seen that

Global sym. in QFT $\Longrightarrow$ Existence of topological operators.

$\underset{?}{\cancel{\Longleftarrow}}$

It's actually sufficent condition for playing the usual game for global sym.

"gauging", "SSB", etc.

$\Downarrow$

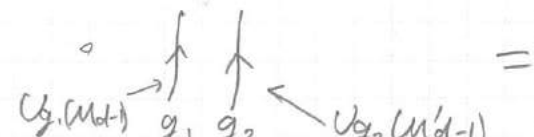
We now regard top. op. as def. of "global sym.", we call it generalized global sym.

From this generalized viewpoint, the (conventional) global sym. can be characterized as follows. (GKSW '14)

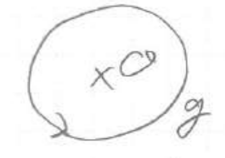
Def. (Conventional global sym.)

d-dim QFT has a global sym. G

$\overset{\text{def.}}{\Longleftrightarrow}$ ◦ For each $M_{d-1} \subset M_d$ and $g \in G$, there is a topological op. $U_g(M_{d-1})$



◦ ← top. op. defined by OPE of U_{g_1} and U_{g_2} should be $U_{g_2 g_1}$

◦ 

$= \times\mathcal{O}' = g\cdot\mathcal{O}$

by accepting higher-codim op.

by accepting more general fusion rule

$\Leftrightarrow$ We have a ~~codim-1~~ $\boxed{\text{top. op.}}$ $U_g(M_{d-1})$ with the ~~group-like~~ fusion rule.

↑ key feature for the sym. (= conservation law)

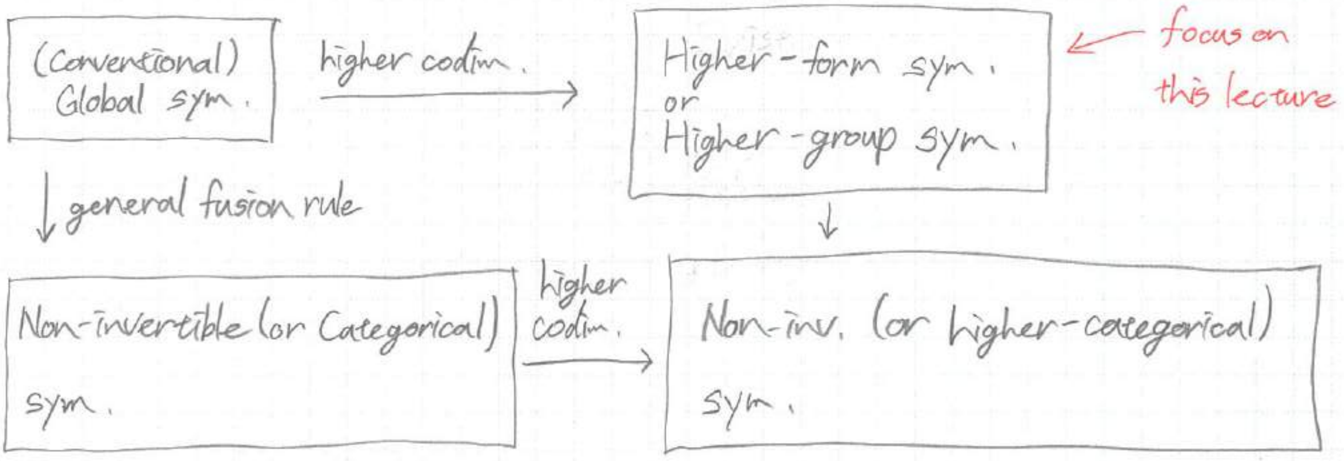
Non-trivial cocycle

$$U_{g_1} \cdot U_{g_2} = \underbrace{e^{i\varphi(g_1,g_2)}}_{\in H^2(BG)} \cdot U_{g_1 g_2}$$

U(1) factor

$\Leftrightarrow$ This sym. G has an 't Hooft anom.

Various directions of generalizations


(Conventional) Global sym.
higher codim.
Higher-form sym. or Higher-group sym.
focus on this lecture
general fusion rule
Non-invertible (or Categorical) sym.
higher codim.
Non-inv. (or higher-categorical) sym.

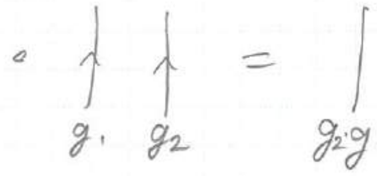
Def. ($\overset{p\text{-form}}{\text{Higher}}$-form sym.)

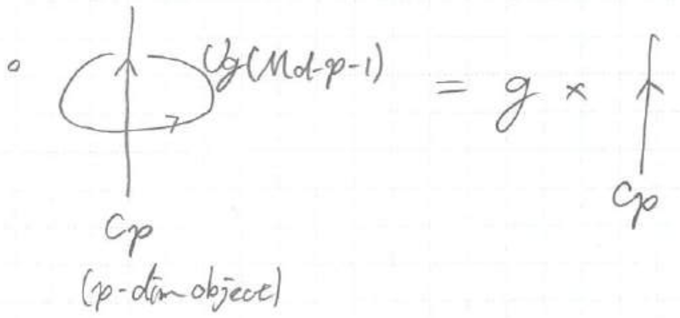
d-dim QFT has a p-form sym. G

$\overset{\text{def.}}{\Longleftrightarrow}$ For each $\overset{\text{codim-}(p+1)\text{ closed mfd}}{M_{d-p-1}} \subset M_d$ and $g \in G$,

there is a topological op. $U_g(M_{d-p-1})$.

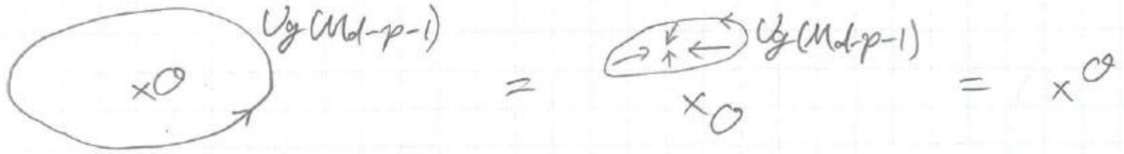
(When we set $p=0$, this def. reduces to the conventional one.)

- 

- 

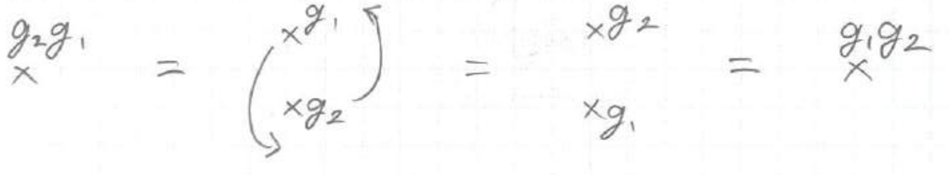
Comment 1.

Higher-form sym. ($p \geq 1$) does not act on local op.



Comment 2.

Algebra of higher-form sym. is Abelian.



$\Rightarrow g_2 g_1 = g_1 g_2$

Examples (Conti and discrete, $G = U(1)$ or $G = \mathbb{Z}_N$)

Ex.1 ($G = U(1)$)

Maxwell theory $\qquad a$: $U(1)$ gauge field

$$S = \frac{1}{2g^2}\int_{M_d} da \wedge {*da}$$

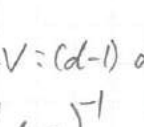
By EoM, $\quad d\big(\underbrace{*\tfrac{1}{g^2}da}_{\text{this is }(d-2)\text{-form}}\big) = 0 \qquad$ EoM takes the form of the conservation law

↓

We can define the codim-2 top. op.

by $\quad U_{e^{i\varepsilon}}(M_{d-2}) = \exp\left(i\varepsilon\int_{M_{d-2}} *\frac{1}{g^2}da\right)$

topologicalness :



$V : (d-1)$ dim $\quad$ s.t. $\partial V_{d-1} = M_{d-2} - M'_{d-2}$

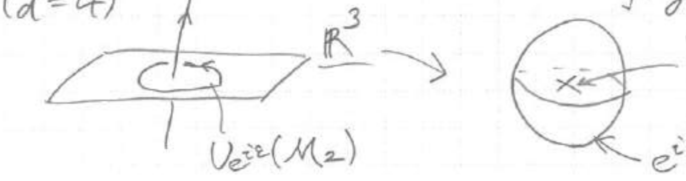
$$U_{e^{i\varepsilon}}(M_{d-2})\, U_{e^{i\varepsilon}}(M'_{d-2})^{-1}$$
$$= e^{i\varepsilon\int_{M-M'} *\frac{1}{g^2}da} \overset{\text{Stokes}}{=} e^{i\varepsilon\int_V \overbrace{d(*\frac{1}{g^2}da)}^{\overset{\text{EoM}}{=}0}} = 1$$

Sym. group is $U(1)$ because $\varepsilon \sim \varepsilon + 2\pi$

Charged op. is given by Wilson loop $\qquad W(C) = e^{i\int_C a}$

this charge has to be ineger by large gauge invariance

$(d=4)$



$\int *\frac{1}{g^2}da = \int \mathbb{E}\cdot d\mathbb{S} = Q = 1$

We're putting electric charge $=1$

$e^{i\varepsilon\int *\frac{1}{g^2}da} = e^{i\varepsilon Q_{ele}} = e^{i\varepsilon}$

This system is often called the electric $U(1)$ 1-form sym.

(We denote it as $U(1)^{[1]}_{\text{ele}}$)

Comment

Maxwell theory has also magnetic sym. followed by Bianchi id.

$$d\left(\frac{1}{2\pi}da\right) = 0$$

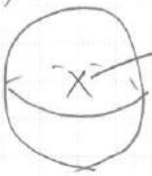
↖ 2-form $\quad 2 = d-(d-2) = d-\underbrace{(d-3)}_{=p}-1$

→ We have $U(1)$ $(d-3)$-form sym.

$$U_{e^{i\varepsilon}}(M_2) = e^{i\varepsilon\int_{M_2}\frac{1}{2\pi}da}$$

It acts on 't Hooft operators defined on C_{d-3}

$(d-3)$



C_0 = p.t. : We remove this point from the spacetime

(i.e. 't Hooft op. is defined as a defect op.)

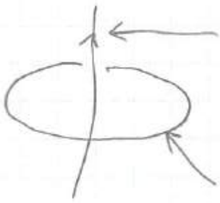
and we put a boundary condition

so that $\int_{S^2}\frac{da}{2\pi} = \underset{\substack{\uparrow \\ \text{magnetic charge}}}{m}\mathbb{Z}$

This sym. is called $U(1)^{[d-3]}_{\text{mag.}}$

Ex2. ($\mathbb{Z}_N$ 1-form sym. of $SU(N)$ YM)

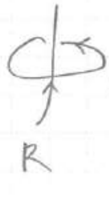
We consider lattice action (We take Wilsonian action)



← $W_R(C) = \mathrm{tr}_R\left[Pe^{i\oint a}\right]$

↖ codim-2 top. op. $U_k(M_{d-2})$

s.t.



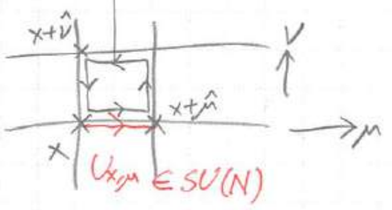
$= e^{\frac{2\pi i}{N}k\cdot\overset{\text{N-ality of rep.}}{|R|}}$

and $U_k \cdot U_{k'} = U_{k+k' \bmod N}$

$$S_W[U_{x,\mu}] = \beta \sum_{x,\mu<\nu} \left\{ \underline{\mathrm{tr}[U_{x,\mu\nu}]} + c.c. \right\}$$

$\beta \sim \frac{1}{g^2} \longrightarrow$ — analogue of hopping term in scalar ca

$$w/ \quad U_{x,\mu\nu} = U^\dagger_{x,\nu} U^\dagger_{x+\hat{\nu},\mu} U_{x+\hat{\mu},\nu} U_{x,\mu}$$



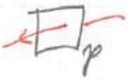
$$U_{x,\mu} \to g^\dagger_{x+\hat{\mu}} U_{x,\mu} g_x$$

Construction of codim-2 defect

We twist the hopping β as follows;

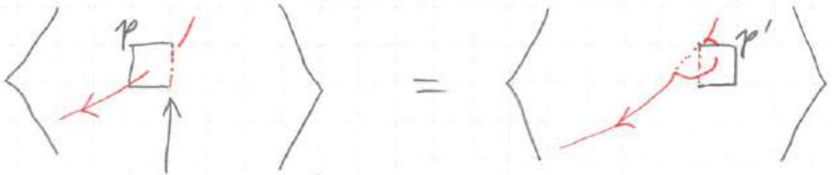
← this is taken on the dual lattice

$$\beta_p = \begin{cases} \beta & \text{if } \Box_p \text{ is not pierced by} \\ \beta \times e^{\frac{2\pi i}{N}k} & \text{if} \end{cases}$$



$$\langle U_k(M_{d-2}) \cdots \rangle = \int DU \, e^{-S_{W,\text{twisted}}[U_{x,\mu}, \beta \to \beta_p]}$$

Theorem $\quad U_k(M_{d-2})$ is topological



$$U_{x,\mu} \xrightarrow[\text{on this link}]{\text{change the dummy variable}} e^{-\frac{2\pi i}{N}k} \cdot U_{x,\mu}$$

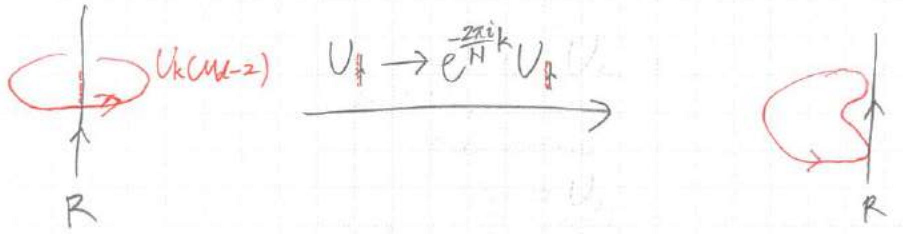
for p, $\quad \beta \times e^{\frac{2\pi i}{N}k} \,\mathrm{tr}\, U_p$

$$\to \beta \times e^{\frac{2\pi i}{N}k} \times e^{-\frac{2\pi i}{N}k} \,\mathrm{tr}\, U_p$$

for p', $\quad \beta \times \mathrm{tr}\, U_{p'}$

$$\to \beta \times \left(e^{-\frac{2\pi i}{N}k}\right)^{-1} \mathrm{tr}\, U_{p'}$$

Commutation relation w/ Wilson loop



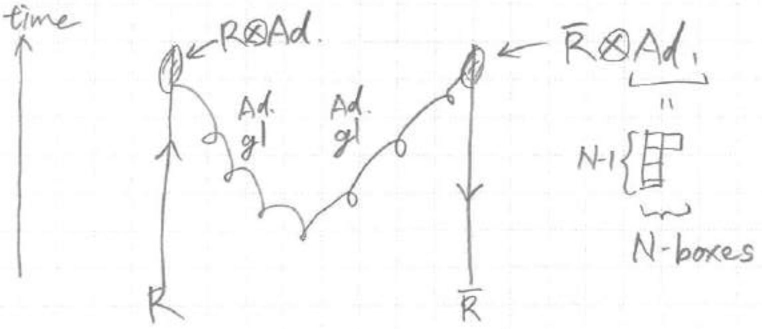
$$W_R(C) \to \mathrm{tr}_R\left(\underbrace{e^{-\frac{2\pi i}{N}k}} P e^{i\oint a}\right)$$

$$= \left(e^{-\frac{2\pi i}{N}k}\right)^{|R|} \mathrm{tr}_R\left(P e^{i\oint a}\right)$$

$\rightsquigarrow$ top. op. $U_k(M_{d-2})$ measures the # of boxes mod N for Wilson loop.

Physical meaning of $\mathbb{Z}_N$ (This is called $(\mathbb{Z}_N^{[1]})_{ele}$)

Classically, electric charge is specified by Rep. $(SU(N))$



$|R| = |R\otimes Ad.|$ ← # of boxes mod N $(\in \mathbb{Z}_N)$ can be a good quantum number.

3コマ目 When we have a $\underbrace{\text{continuous global sym. } G}$, ← j_a : Noether current

we can consider the partition function w/ the background G-gauge field A :

$$Z[A] = \int \mathcal{D}\Phi\, e^{-S[\Phi] + \underbrace{i\int A\wedge j}_{} + \mathcal{O}(A^2)}$$ ← minimal coupling term

$$\left(\text{Naively, } Z[A+d\lambda] = \left\langle e^{i\int (A+\underbrace{d\lambda)\wedge j}_{=\lambda\wedge \underbrace{dj}_{=0}} + \mathcal{O}(A^2)} \right\rangle = Z[A]\right)$$

Z[A]: Generating functional for correlators of Noether currents

$$\frac{\delta^n Z}{\delta A(x_1)\cdots\delta A(x_n)} = \langle j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n)\rangle + \text{correlator including } A$$

We'd like to consider Z[A] also for generalized sym. $\left\{\begin{array}{l} G: \text{discrete/} \\ \text{higher-form/} \\ \text{higher-group}\end{array}\right.$

Symmetry $\Leftarrow$ topological operators

Background gauge fields $\overset{\text{def}}{:=}$ Network of topological op.

(↑ I'd like to give a convincing arg. for its validity.)

Ex.1. (Scalar theory w/ sym. G on lattice)

$$S = \sum_{\langle x,x'\rangle} J\, \varphi^\dagger_{(x)}\cdot\varphi(x') + \sum_x V(\varphi(x), \varphi^\dagger_{(x)})$$

$g \in G$ acts as $\varphi(x) \to g^\dagger \varphi(x)$ and $V(g^\dagger\varphi, (g^\dagger\varphi)^\dagger) = V(\varphi, \varphi^\dagger)$

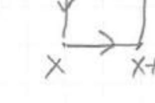
Now we introduce the background G-gauge field $\underset{U_{x,\mu}\in G}{\overset{\text{link variable}}{\swarrow}}$

$$Z[\{U_{x,\mu}\}] = \int D\varphi\, e^{-S[\varphi(x), U_{x,\mu}]}$$

w/ $S[\{\varphi(x)\}, \{U_{x,\mu}\}]$

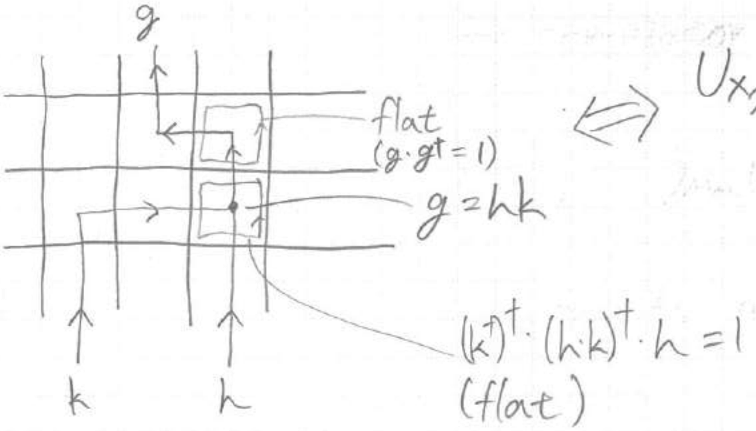
$$= \sum_{x,\mu}\left(\varphi^\dagger_{(x+\hat\mu)} U_{x,\mu}\varphi(x) + \varphi^\dagger_{(x)} U^\dagger_{x,\mu}\varphi(x+\hat\mu)\right) + \sum_x V(\varphi(x), \varphi^\dagger_{(x)})$$

When G is discrete, let's restrict our attention to the $\underset{\Updownarrow\, F=0}{\text{"flat"}}$ gauge field.

(i.e. 

$$U_{x,\mu\nu} = U^\dagger_{x,\nu} U^\dagger_{x+\hat\nu,\mu} U_{x+\hat\mu,\nu} U_{x,\mu} = 1 \in G$$)

Let's draw an example of such a configs ;

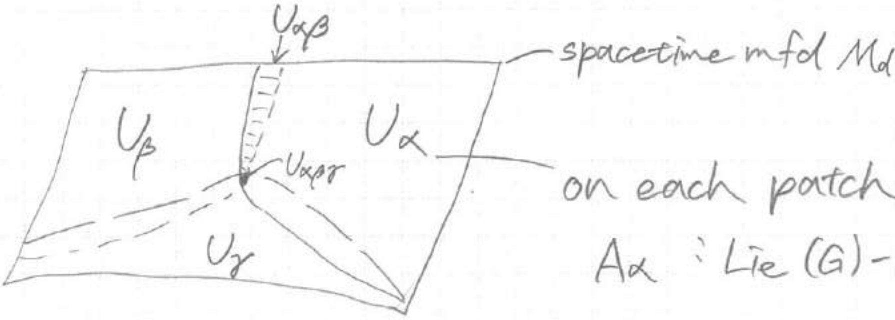


$\Longleftrightarrow$ $U_{x,\mu} = \begin{cases} 1 & \text{if } x \text{---} x+\hat{\mu} \text{ is not crossed by anyth} \\ g_{(=hk)} & \text{if } x \underset{g}{\uparrow} x+\hat{\mu} \\ h & \text{if } x \underset{h}{\uparrow} x+\hat{\mu} \\ k & \text{if } x \underset{k}{\uparrow} x+\hat{\mu} \end{cases}$

↑ This is an example of the network of top. operators

Let me give its continuum interpretation ;

G-gauge field $\Leftarrow$ Connection of G-bundle over the spacetim



on each patch U_α ;
A_α : Lie (G)-valued 1-form

on $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $g_{\alpha\beta} : U_{\alpha\beta} \to G$ (transition function)

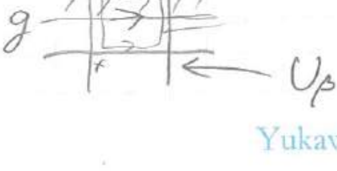
w/ the connection formula

$$A_\beta|_{U_{\alpha\beta}} = g_{\alpha\beta}^\dagger (A_\alpha|_{U_{\alpha\beta}}) g_{\alpha\beta} + g_{\alpha\beta}^\dagger d g_{\alpha\beta}$$

on $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, we impose the cocycle condition

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \qquad (g_{\beta\alpha} = g_{\alpha\beta}^\dagger)$$

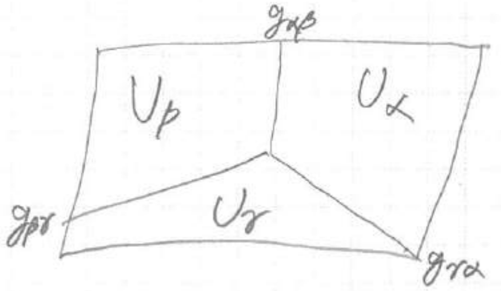
※ on the lattice, $\{U_{x,\mu}\}$ contain the both data of $\{A_\alpha\}$ and $\{g_{\alpha\beta}\}$



For simplicity, let's again restrict our attention when G is descrete.

$\rightsquigarrow A_\alpha = 0$

So all data of G-bundle is carried by $g_{\alpha\beta}$.



$g_{\alpha\beta} = g_{\alpha\gamma} g_{\gamma\beta}$ ← nothing but the flatness condition for the network.

Also for discrete sym.

A = network of top. G-defects.

(= specification of the transition function of the G-bundle,

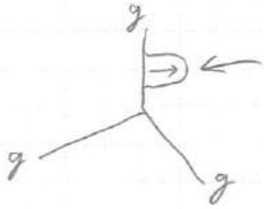
$$Z[A] = \left\langle \left(\begin{matrix} \text{corresponding} \\ \text{network} \end{matrix} \right) \right\rangle$$

It's now easy to generalize it to discrete higher-form sym.

Background gauge field for $\mathbb{Z}_N^{[1]}$,

B = Network of codim-2 top. surfaces.

↳ $Z[B]$ ← partition func. computed with the twisted b.c. specified by the network.



← This kinds of deformation

⇔ gauge transformation for background gauge fields

Let's try to compute Z[background gauge field]

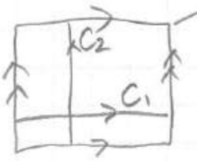
for several simple examples.

Let's consider the Ising model in the ferro state.

$$S[\underset{\substack{\uparrow \\ \varphi=\pm 1}}{\varphi(x)}] = -J\sum_{\langle x,x'\rangle}\varphi(x)\varphi(x')$$

$$Z[\underbrace{A}_{\substack{U(x,x')=\pm 1\\ \text{flatness}}}] = \sum_{\{\varphi(x)\}=\{\pm 1\}} e^{+J\sum_{\langle x,x'\rangle}\varphi(x)\,U(x,x')\,\varphi(x')}$$ ← This is manifesely background gauge inv.

$\Rightarrow \delta(A)$ ← A has to be trivial up to gauge transformation



spacetime $= T^2$

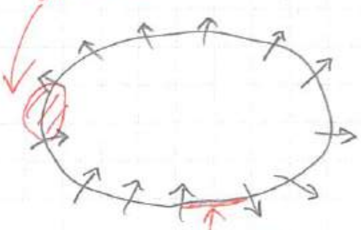
→ gauge inv. data

$$\Leftrightarrow \int_{C_{1,2}} A = 0 \text{ or } \pi$$

$$\prod_{\langle x,x'\rangle\in C_i}\underbrace{U(x,x')}_{\pm 1} \in \{\pm 1\}\simeq \mathbb{Z}_2$$

If we take $\int_{C_1} A = \pi$ (non-trivial twist along x_1 direction)

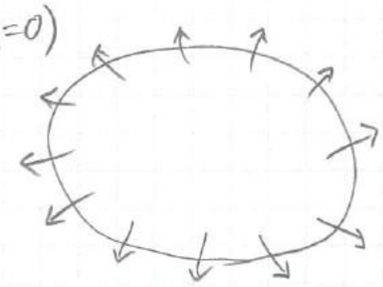
domain-wall is created.



$U(x,x') = -1$

$\Leftrightarrow J(x,x') = -J$ (anti-ferro)

$(A=0)$



$\Leftrightarrow J = +J$ (ferro)

$$\frac{Z_{\int A=\pi}}{Z_{\int A=0}} = e^{-L_{time}\times T_{domain\text{-}wall}} \xrightarrow{\substack{L_{time}\\ L_{space}\to\infty}} 0$$

Theorem $G^{[0]}$ is global sym.

$$G\xrightarrow{SSB} H \iff Z[A]\xrightarrow{V\to\infty}\overbrace{\delta([A]\in H^1(M,H))}^{=\frac{1}{Vol(gauge)}\sum_{[a]\in H^1(M,H)}\delta(A-a)}$$

(To circumvent the suppression, A has to be the gauge field for unbroken sym.)

$$Z = e^{-L_{time} \times E_{G.S.}[A]}$$
↖ twisted b.c.

For non-twisted case, $E_{G.S.}$ = ground state energy for ferro.

For twisted, $E_{G.S.} = \overset{\shortparallel}{\int_x \varepsilon_{ferro}}$ + domain wall tension

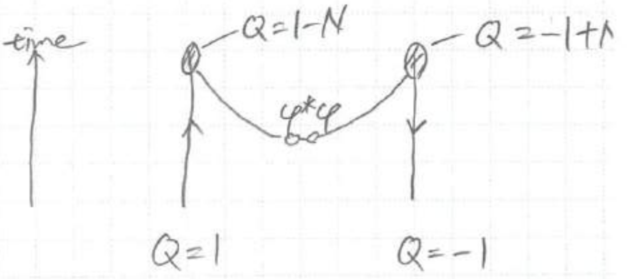
Ex. Example of SSB for higher form sym. : $\mathbb{Z}_N^{[1]} \xrightarrow{SSB} 1$

We can easily realize it by considering the Abelian-Higgs model.

$$\underset{\substack{\nearrow \\ \text{cpx scalar}}}{S[\varphi}, \underset{\substack{\uparrow \\ U(1)\text{ gauge}}}{a]} = \int |(d + i\underset{\substack{\uparrow \\ \text{charge-}N}}{N}a)\varphi|^2 + V(|\varphi|^2)$$

This model has $\mathbb{Z}_N^{[1]}$ sym.

If we consider Q in mod N, this is not changed by the screening of φ, φ^* quanta. $\mathbb{Z}_N^{[1]}$ is the best sym. that we could imagine.



- $U_{x,\mu} = e^{i a_{x,\mu}} \in U(1)$
- $\varphi(x) \in \mathbb{C}$
- $U_{x,p} = e^{iF_{x,\mu\nu}}$ is $U(1)^{[1]}$ inv. ↙ Maxwell theory
- $|D_\mu\varphi|^2 \simeq \varphi^\dagger_{(x+\hat\mu)} (U_{x,\mu})^{N} \varphi_{(x)}$

$$x \underset{U^\dagger_{x,\mu}}{\overset{U_{x,\mu}}{\rightleftarrows}} x+\hat\mu$$

$$U_{x,\mu} \longrightarrow \underbrace{e^{-\frac{2\pi i}{N}}}_{\in} U_{x,\mu}$$
$$\underbrace{\mathbb{Z}_N^{[1]} \subset U(1)^{[1]}}$$
↑
comes from the hopping ter

→ This lattice construction give the justification of $\mathbb{Z}_N^{[1]}$.

$$Z[B] = \int \mathcal{D}b\, \mathcal{D}a \, \exp\left(i\frac{N}{2\pi} \int b \wedge (da + B) \right)$$
$$= \delta(B) \rightsquigarrow \mathbb{Z}_N^{[1]} \xrightarrow{SSB} 1$$

Let's derive the IR EFT $\quad \varphi = v \times e^{i\varphi}$ ↖ We can focus on the U(1)-value phase factor

↳ $S_{eff} = \frac{v^2}{2} \int |d\varphi - Na|^2 \overset{v^2 \to \infty}{\Longleftrightarrow} S_{top} = i\frac{N}{2\pi} \int b \wedge da$

10/17 Review of 1st day

(Generalized) Symmetry = Topological operators

$U(M_{d-p}) = U(M'_{d-p})$ ← as an op. identity

continuous deformable each other

Conventional sym. → higher-form sym. ← as an example, gauge theories often have 1-form sym.

↓ ↓ ↑ "center" sym.

non-inv. sym → higher cat. sym.

For group like sym. (including 0-form / higher form cases)

Network of $U_g(M_{d-p})$ ⇒ background gauge fields

top. op. →

G-bdle over spacetime mfd

- A_α : local diff form
- $g_{\alpha\beta}$: transition func.

G-sym. QFT ⇐ "a functor from Bord. cat. with G-str. to Vector cat."

$$Z[M] = Z[A] \in \mathbb{C}$$

$Z[A]$ nicely capture the vacuum realization of sym.

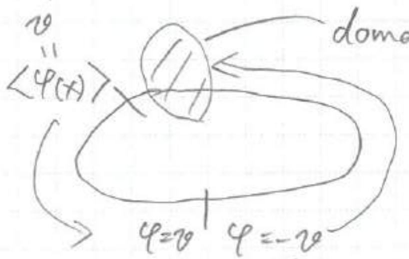
$G^{[0]}$ ← finite

$$\rightsquigarrow \quad G \xrightarrow{SSB} H$$

$$Z[A] \xrightarrow{\text{low energy}} \sum_{a \in H^1(M;H)} \delta(A-a)$$

$$\varphi(x) \cdot P e^{i\int_x^y A} \varphi^\dagger(y)$$

↑ order parameter : this is in a non-trivial irrep. of G



domain wall must be created somewhere

↑ Aharanov-Bohm phase by background gauge field

Example for SSB of 1-form sym. (昨日の続き)

charge-N Abelian-Higgs model

$$S[\varphi,a]=\int |(d+i\underbrace{N}_{\mathbb{Z}_{>0}}a)\underset{\uparrow\text{ cpx scalar}}{\varphi}|^2+V(|\varphi|^2)+\frac{1}{2g^2}|d\underset{\substack{\uparrow\\ U(1)\text{ gauge field}}}{a}|^2$$

$$\left(\begin{array}{l}U(1)\text{ transformation is}\quad a\to a+d\underset{\uparrow\,2\pi\text{-periodic scalar}}{\lambda}\\ \qquad\qquad\qquad\qquad\quad \varphi(x)\to e^{-iN\lambda(x)}\varphi(x)\end{array}\right)$$

This model has $\mathbb{Z}_N^{[1]}$ sym.

$$\mathbb{Z}_N^{[1]}\subset U(1)^{[1]}\leftarrow\text{1-form sym. of Maxwell}$$

We can introduce the network of codim-2 topological defects.

$\uparrow$ $\mathbb{Z}_N$ 2-form gauge field B

Comment : In general, for p-form sym. the corresponding b.g. gauge field is regarded as $(p+1)$-form field.

For 0-form case,

$\nwarrow$ $(d-1)$-dim surface = codim-1 top. defect

In general,

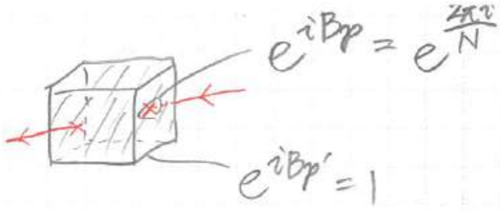
$$p\text{-form}\xleftrightarrow{\text{Poincare duality}}\underset{\substack{\uparrow\\ \text{constructed by taking the transverse}\\ \text{direction for }A_{\mu_1\cdots\mu_p}}}{(d-p)}\text{-dim surface}$$

We can consider lattice realization to see this more explicitly

$$S=\sum_{p:\text{plaquette}}\frac{1}{g^2}e^{iB_p}\underbrace{U_p}_{\substack{\shortparallel\\ \prod_{\ell\in p}U_\ell}}+\sum_{\substack{\ell=\langle x,x'\rangle\\ (\text{link})}}\left(\varphi^\dagger(x)(U_\ell)^N\varphi(x')+\text{c.c.}\right)$$

$$+\sum_{x:\text{sites}}V(|\varphi(x)|^2)$$

$$e^{iB_p}\in\left\{1,e^{\frac{2\pi i}{N}},\cdots,e^{\frac{2\pi i}{N}(N-1)}\right\}$$: $\mathbb{Z}_N$ phase factors defined on plaquettes.



$e^{iB_p} = e^{\frac{2\pi i}{N}}$

$e^{iB_{p'}} = 1$

When Higgs potential is deep enough, $V =$

then $Z[B] = \delta(B)$ in IR limit

$\rightsquigarrow \mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} 1$

To see this, let's take the IR limit of cherge-N Abelian-Higgs model. $(d \geq 3)$

$$V(|\varphi|) = \lambda(|\varphi|^2 - v^2)^2 \qquad \overset{\text{deep IR}}{\overset{\downarrow}{\lambda \gg 1}}$$

$\Rightarrow$ Classical vacua can be parametrized as

$$\varphi(x) = v \cdot e^{i\underbrace{\phi(x)}_{\leftarrow \text{ } 2\pi\text{-periodic scalar field}}}$$

Neglect the radial motion at the tree-level

$$S_{\text{eff}}[\phi, a] = \int \underbrace{v^2}_{\uparrow} |d\phi + Na|^2 + \cancel{\left(\frac{1}{g^2}|da|^2\right)}$$

$[v] = [\varphi] = \frac{d-2}{2}$ ← for $d \geq 3$, v has mass dim.

↑ gauge field has $m^2 \sim g^2 v^2 N^2$

With RG trans. $v \xrightarrow{\text{IR limit}} \infty$

In order to take $v \to \infty$ correctly, we introduce an auxiliary field ;

$\frac{1}{v^2}|h + v^2 i * (d\phi + Na)|^2 + \frac{1}{v^2}|iv^2 * (d\phi + N$

$$e^{-S_{\text{eff}}[\phi, a]} = \int \mathcal{D}h \, e^{-\left(\int \frac{(2\pi^2)}{v^2} h \wedge * h + i\frac{1}{2\pi}\int h \wedge (d\phi + Na)\right)}$$

? ↑

v^2 appear in the denominator $\Rightarrow$ in $v \to \infty$ limit,

this term drops.

$$\left(\text{EoM.} \quad h = v^2 * (\underbrace{d\phi}_{\uparrow \ \mathbb{R}\text{-valued}} + Na)\right)$$

Therefor, IR effective partition function is given by

$$Z = \int \underbrace{\mathcal{D}h}_{\substack{\mathbb{R}\text{-valued} \\ (d-1)\text{-form field}}} \underbrace{\mathcal{D}\phi}_{\substack{2\pi\text{-periodic} \\ \text{scalar}}} \underbrace{\mathcal{D}a}_{U(1)\text{ gauge field}} \; e^{i\frac{1}{2\pi}\int h\wedge(d\phi+Na)}$$

This action does not contain any dimensionful parameters.
This is an example of TQFT.

Let's study the behaviors of this effective theory.

EoM of h : $d\phi + Na = 0$

Wilson loop obeys the perimeter la

$$\Rightarrow \underset{e^{i\int a}}{\underset{\shortparallel}{\langle W(a)\rangle}} = \langle e^{i\frac{1}{N}\int_C d\phi}\rangle = \begin{cases} 1 & \text{if } C \text{ is contractib} \\ \underbrace{e^{\frac{2\pi i}{N}k}}_{\text{This can take } N \text{ distinct values.}} & \text{if } C \text{ is not contractible.} \end{cases}$$

Def. (SSB of 1-form sym.)

$W(C)$: charged object of $\mathbb{Z}_N^{[1]}$ sym.

$$\langle W(C)\rangle \sim \begin{cases} e^{-\sigma\times \mathrm{Area}(C)} & \text{area law} \iff \mathbb{Z}_N^{[1]} \text{ is unbroken} \\ e^{-\mu\times \mathrm{Length}(C)} & \text{perimeter law} \iff \mathbb{Z}_N^{[1]} \text{ is broken} \end{cases}$$

Possible IR behaviors of loop op. in gapped phase

comment : This def. of SSB can be regarded as the higher form generalization of SSB by ODLRO

$$\langle \varphi^*(x)\varphi(y)\rangle_{\text{conn.}} \xrightarrow{|x-y|\to\infty} \begin{cases} e^{-\frac{|x-y|}{\xi}} \to 0 & \text{unbroken sym.} \\ v^2 \neq 0 & \text{broken sym} \end{cases}$$

Let's perform computation of $Z[B]$ (partition func. w/ b.g. gauge field) ;

To do it, let's derive another form of the EFT.

Instead of integrating out h first, we perform the path int. of ϕ at the beginning.

$$\text{EoM of } \varphi \ : \ d\left(\frac{1}{2\pi}h\right) = 0 \quad ; h \text{ is a closed } (d-1)\text{-form}$$

$$\rightarrow h = db \quad \text{(at least locally)}$$

Globally, it turns out that $h = db$ w/ b : $U(1)$ $(d-2)$-form gauge field

Rough idea : $\phi \sim \phi + 2\pi$ redundancy

$$\underset{\mathbb{R}\text{-valued field}}{\phi = \hat{\phi}} + \sum_{i=1}^{d} 2\pi \underbrace{n_i}_{\in \mathbb{Z}} \frac{x_i}{L_i}$$

spacetime $= T^d$

$$\int \mathcal{D}\phi := \sum_{i\in\mathbb{Z}} \int \mathcal{D}\hat{\phi}$$

after $\int \mathcal{D}\hat{\phi}$, $\sum_{\{n_i\}}$ requires that

$$\frac{1}{2\pi} \int_{\underbrace{M_{d-1}}_{\text{closed } (d-1)\text{-dim submfd.}}} \overbrace{h}^{=db} \in \mathbb{Z} \Rightarrow \text{Dirac quantization condition}$$

Therefore,

$$Z = \int \underset{U(1)\ (d-2)\text{-form gauge field}}{\mathcal{D}b}\ \underset{U(1)\ 1\text{-form gauge field}}{\mathcal{D}a}\ e^{\underbrace{i \overset{\text{gauge charge of higgs}}{N} \frac{1}{2\pi} \int b \wedge \underbrace{da}_{f}}_{\text{BF theory}}}$$

In this IR description, we can write down the explicit form of the $\mathbb{Z}_N^{[1]}$ sym. generators.

$$U_k(M_{d-2}) = e^{ik\int_{M_{d-2}} b} \qquad \text{(Wilson op. for } b\text{)}$$

- Topologicalness

EoM of a : $db = 0$

$$U_k(M_{d-2}) = U_k(M_{d-2}) \cdot e^{ik\int_V \overset{0}{\overset{\shortparallel}{db}}} = U_k(M'_{d-2})$$

- $\mathbb{Z}_N$ fusion rule

$U_k U_{k'} = U_{k+k'}$ is obvious

$U_k \in \mathbb{Z}_N$ is non-trivial

$$e^{i \underbrace{\int \frac{Nb}{2\pi}}_{\overset{\cap}{\mathbb{Z}}} \underbrace{\int da}_{\in 2\pi\mathbb{Z}}}$$

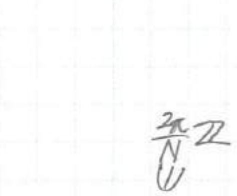
$$\underbrace{\int_{M_{d-2}}}_{\text{closed}} \frac{b}{2\pi} \in \frac{1}{N}\mathbb{Z} \quad \rightsquigarrow \quad U_k(M_{d-2}) = e^{ik \overbrace{\int b}^{\overset{\frac{2\pi}{N}\mathbb{Z}}{\cup}}} = e^{2\pi i \frac{k \times \overset{\text{int}}{\downarrow}\#}{N}} \in \mathbb{Z}$$

Now, we can easily evaluate $Z[\underset{\uparrow}{B}]$

b.g. gauge field for $\mathbb{Z}_N^{[1]}$
$\shortparallel$
network of sym. defect

$M_2 \subset M_d$ ← 2-cocycle of spacetime

$\int_{M_2} B = \frac{2\pi}{N}$: non-trivial flux



$$\frac{Z[B]}{Z[0]} = \left\langle e^{i\int_{M_{d-2}} b} \right\rangle$$

↑ transverse $(d-2)$-dim

$$= \int \mathcal{D}b\,\mathcal{D}a \; e^{i\frac{N}{2\pi}\int b\wedge da + i\int_{M_{d-2}} b}$$

$$= \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}k} = 0$$

Other expression

B : 2-form gauge field

1-form gauge trans.

$$\begin{cases} B \to B + d\Lambda \\ a \to a + \Lambda \end{cases}$$

$$Z[B] = \int \mathcal{D}a\,\mathcal{D}b \; e^{\frac{iN}{2\pi}\int b\wedge(da - B)} \rightsquigarrow \text{EoM of } b : B = \underset{\sim\,=0}{da} \quad \text{(i.e. } B \text{ is in a pure gaug}$$

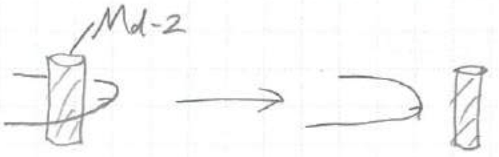
$\to Z[B] = \delta(B)$

2コマ目

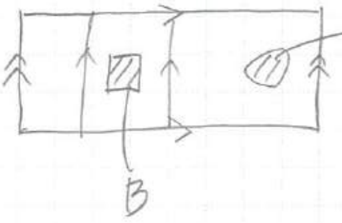
Physical interpretation of $\delta(B)$

$W(C) = e^{i\int_C a}$: Wilson loop

$U(M_{d-1}) = e^{i\int b}$: sym. generator



$\times e^{\frac{2\pi i}{N}}$



surperconducting vortex must exist somewhere

$$Z[\text{(non-trivial) } B] = e^{-\overset{\text{length of transverse direction}}{L} \times (\text{vortex energy})}$$

$$\xrightarrow{Vol\to\infty} 0$$

We can recover the SSB of $\mathbb{Z}_N^{[1]}$ defined by perimeter law by $\delta(B)$-behavior of $Z[B]$.

Application of these techniques for SU(N) gauge theories

- Give continuum description of $\mathbb{Z}_N$-gauging (Kapustin-Seiberg '14)
- Derive an 't Hooft anomaly between $\mathbb{Z}_N^{[1]}$ and $\theta\sim\theta+2\pi$ of 4d YM (TTT '17)
- Re-derivation of Wilson-'t Hooft criterion (Nguyen, Tanizaki, Ünsal '23)

4d SU(N) YM + Adj. matter

$\uparrow$ electric $\mathbb{Z}_N^{[1]}$ sym. (This is seen in Wilsonian lattice)

I'm here going to give a continuum description for codim-2 top. networks B.

Preliminary example

φ : $\mathbb{C}$-valued scalar

$$S = \int |d\varphi|^2 + \lambda(\varphi^N + \varphi^{*N})$$

$\mathbb{Z}_N$ sym. $\varphi \to e^{\frac{2\pi i}{N}}\varphi$ $\quad (\subset U(1) : \varphi \to e^{i\alpha}\varphi)$

→ Let's introduce b.g. $\mathbb{Z}_N$ gauge field

$\mathbb{Z}_N$ gauge field seems to be a special case of $U(1)$ gauge field.
Let's realize it as a pair of $U(1)$ gauge field and 2π-periodic scalar.

$$A_{\mathbb{Z}_N} = (\underset{\substack{\uparrow \\ U(1)\text{gauge} \\ \text{field}}}{A}, \underset{\substack{\uparrow \\ 2\pi\text{ periodic} \\ \text{scalar}}}{\phi}) \qquad \text{w/} \quad NA = d\phi \ \leftarrow \text{This appears in one of IR description of Abelian-Higgs model.}$$

We can couple (A, ϕ) so that $U(1)$ gauge inv. under

$$\begin{cases} A \to A + d\overset{\swarrow 2\pi \text{ periodic scalar}}{\lambda} \\ \phi \to \phi + N\lambda \\ \varphi \to e^{-i\lambda}\varphi \end{cases}$$

$$S[\varphi, (A,\phi)] = \int |(d+iA)\varphi|^2 + g(e^{i\phi}\varphi^N + e^{-i\phi}\varphi^{*N})$$

Fact Any network for $\mathbb{Z}_N^{[0]}$ can be realized as (A, ϕ) with this way of coupling.

Idea $NA = d\phi \rightsquigarrow dA = 0$ i.e. A is flat.

Flat $U(1)$ bdle can be completely specified by transition functions.

This example suggests that $\mathbb{Z}_N$ gauge fields accepts the continuum description by embedding $\mathbb{Z}_N \subset U(1)$
In general, $\mathbb{Z}_N$ k-form gauge field is realized by a pair of $U(1)$ k-form and (k-1)-form gauge fields $(B^{(k)}, B^{(k-1)})$, with the constraint $NB^{(k)} = dB^{(k-1)}$.

4d SU(N) YM coupled w/ $\mathbb{Z}_N$ 2-form gauge field

$$\text{A pair } (\underset{\substack{\nearrow \\ \text{U(1) 2-form} \\ \text{gauge field}}}{B}, \underset{\substack{\uparrow \\ \text{U(1) 1-form} \\ \text{gauge field}}}{C}) \quad \text{w/} \quad NB = dC$$

We now need to consider how these fields couple to SU(N) gauge field a $\longrightarrow$ embed it into $U(N)$ gauge field $\tilde{a}$

$$\text{“}\tilde{a} \overset{\text{locally}}{=} \underset{\updownarrow}{a} + \frac{1}{N} \underset{\updownarrow}{C}\, \mathbb{1}_{N\times N}\text{”}$$

$$U(N) = \frac{SU(N) \times U(1)}{\mathbb{Z}_N}$$

We need to specify 1-form gauge transformation and achieve its invariance.

$$\begin{cases} B \to B + d\Lambda \;\leftarrow \text{U(1) 1-form gauge field} \\ C \to C + \underline{N\Lambda} \;\leftarrow \text{This is specified by } NB = dC \\ \tilde{a} \to \tilde{a} + \Lambda \mathbb{1}_N \end{cases}$$

SU(N) field strength $\quad f = da + i a \wedge a$

$\downarrow$

U(N) field strength $\quad \hat{f} = d\tilde{a} + i \tilde{a} \wedge \tilde{a}$

Under U(1) 1-form gauge trans.

$$\hat{f} \to d(\tilde{a} + \Lambda) + i \underbrace{(\tilde{a} + \Lambda) \wedge (\tilde{a} + \Lambda)}_{\substack{\shortparallel \\ \tilde{a} \wedge \tilde{a} + \underbrace{\Lambda \wedge \tilde{a} + \tilde{a} \wedge \Lambda}_{=0} + \underbrace{\Lambda \wedge \Lambda}_{=0}}}$$

$$= \hat{f} + d\Lambda$$

We can now construct $U(1)^{[1]}$-inv. action of a

$$S[\tilde{a}, (B, C)] = \int \frac{1}{g^2} \mathrm{tr}\left[(\hat{f} - B) \wedge * (\hat{f} - B)\right] + \frac{i\theta}{8\pi^2} \int \mathrm{tr}\left[(\hat{f} - B) \wedge (\hat{f} - B)\right]$$

$$Z[(B,C)] = \underbrace{\int \mathcal{D}\tilde{a}}_{\uparrow} \, \delta(\mathrm{tr}[\tilde{a}]-C)\, e^{-\int \frac{1}{g^2}\mathrm{tr}[(\tilde{f}-B)\wedge *(\tilde{f}-B)] + \frac{i\theta}{8\pi^2}\int \mathrm{tr}[(\tilde{f}-}$$

We're summing up all $U(N)$ gauge fields $\tilde{a}$ w/ the constraint $\underline{\mathrm{tr}[\tilde{a}] = C}$

Comments on the global str. of gauge grp.

(cf. Aharony, Seiberg, Tachikawa '13)

$$\underset{\uparrow}{SU(N)} \text{ v.s. } \frac{SU(N)}{\mathbb{Z}_N} \text{ v.s. } \frac{SU(N)}{\underset{\underset{n|N}{\uparrow}}{\mathbb{Z}_n}} \cdots$$

If you work on YM + fund. matter, $SU(N)$ is the unique option

If you work on YM + adj. matter, the local properties do not specify the global str.

Let's start from $SU(N)$ YM $\leftarrow \mathbb{Z}_N^{[1]}$

$\Downarrow$

$Z[(B,C)]$ ($\mathbb{Z}_N$ 2-form b.g. gauge field)

We can perform the path int. for (B,C) to obtain a new QFT.

$$\underbrace{Z_{\text{new QFT}}}_{SU(N)/\mathbb{Z}_N \text{ YM}} = \int \mathcal{D}b\mathcal{D}c\, \delta(Nb-dc)\, Z_{YM}[(b,c)]$$

SU(N) YM

$W(C)$: Wilson loop op. for test electric particle.

$\underbrace{H(C,S)}_{C=\partial S}$: 't Hooft loop op. (non-genuine) for test magnetic particle.

$\uparrow$ This operator "secretly" depends on a surface s.t. $\partial S = C$

$\underline{\frac{SU(N)}{\mathbb{Z}_N} \text{ YM}}$ $\leftarrow (\tilde{a}, b, c)$ $Nb = dc$, $\mathrm{tr}(\tilde{a}) = c$

$$\underbrace{W(C,S)}_{} = W(C)\, e^{-i\int_S b} \quad ; \text{ Wilson loop op. (non-genuine)}$$

$$\hookrightarrow \mathrm{tr}(P e^{i\int_C \tilde{a}})$$

$H(C)$: 't Hooft loop op.

$\circlearrowright$

$\mathbb{Z}_N^{[1]}$ mag.

In general, gauging of $\mathbb{Z}_N^{[p]}$ in d-dim QFT $\mathcal{T}$,
it produces another d-dim QFT "$\mathcal{T}/\mathbb{Z}_N^{[p]}$"
which has "dual" $\mathbb{Z}_N^{[d-p-2]}$ sym.

Simplest e.g. 2d classical Ising : $\mathbb{Z}_2^{[0]}$

$$\text{Ising}/\mathbb{Z}_2^{[0]} \underset{\substack{\uparrow \\ \text{Kramers–Wannier duality}}}{\simeq} \text{Ising}$$

Proof

$$Z_{\mathcal{T}}[\underbrace{B^{(p+1)}}_{=(B^{(p+1)},\, B^{(p)}) \text{ w/ } NB^{(p+1)} = dB^{(p)}}] \quad \swarrow \mathbb{Z}_N^{[p]} \text{ gauge field}$$

$$Z_{\mathcal{T}/\mathbb{Z}_N^{[p]}} = \int \mathcal{D}b^{(p+1)} \mathcal{D}b^{(p)}\, \delta(Nb^{(p+1)} - db^{(p)})\, Z_{\mathcal{T}}[(b^{(p+1)}, b^{(p)})]$$

$$= \int \mathcal{D}h^{(d-p-1)} \mathcal{D}b^{(p+1)} \mathcal{D}b^{(p)}\, \underset{\substack{\uparrow \\ \text{EoM of } h \,:\, Nb^{(p+1)} = db^{(p)}}}{e^{\frac{i}{2\pi}\int h^{(d-p-1)} \wedge (Nb^{(p+1)} - db^{(p)})}} \cdot Z_{\mathcal{T}}[(b^{(p+1)}, b^{(p)})]$$

$$\Rightarrow \underline{db^{(p+1)} = 0}$$

$$\tilde{U}(\underbrace{M_{p+1}}_{d-(d-p-2)-1}) = e^{i\int_{M_{p+1}} b^{(p+1)}} \text{ is a top. op.} \longleftarrow$$

$\Rightarrow$ This is a generator of $(d-p-2)$-form sym. of $\mathcal{T}/\mathbb{Z}_N^{[p]}$

$$Z_{\mathcal{T}/\mathbb{Z}_N^{[p]}}[(B^{(d-p-1)}, B^{(d-p-2)})] \quad \longleftarrow \mathbb{Z}_N^{[d-p-2]} \text{ b.g. gauge field}$$

$$\downarrow \int \mathcal{D}B^{(d-p-1)} \mathcal{D}B^{(d-p-2)}$$

$$Z_{(\mathcal{T}/\mathbb{Z}_N^{[p]})/\mathbb{Z}_N^{[d-p-2]}} = Z_{\mathcal{T}} \quad \mathbb{Z}_N^{[p]} \text{ in original theory.}$$

$$\hookrightarrow \text{dual } (\underset{\substack{\| \\ p}}{d-(d-p-2)-2})\text{-form sym.} \updownarrow$$

10/19 To study possible quantum phases of 4d QFT w/ $\mathbb{Z}_N^{[1]}$ sym.

(in particular, 4d SU(N) YM + Adj. matter)

We canconsider $Z[B]$: partition func. w/ $\underline{\text{bg. gauge field.}}$

$\uparrow$ $\mathbb{Z}_N^{[1]}$ gauge field.

$\Leftarrow$ a pair of U(1) 2-form and 1-form gauge fields (B, C)
w/ $NB = dC$

For 4d SU(N) YM,

$$SU(N) \text{ gauge field } a \overset{\text{promote}}{\Longrightarrow} U(N) \text{ gauge field } \tilde{a} = a + \frac{1}{N} C \mathbb{1}_N$$

$$B \to B + d\Lambda, \quad C \to C + N\Lambda$$
$$\tilde{a} \to \tilde{a} + \Lambda \mathbb{1}_N$$

$$Z_{SU(N)YM}[B] = \int \mathcal{D}\tilde{a}\, \underset{\substack{\Updownarrow \\ tr(\tilde{f}) = NB}}{\delta(tr(\tilde{a}) = C)}\, e^{-S_{YM}\underbrace{[\tilde{a}, B]}_{\substack{\uparrow \\ \text{contained by performed} \\ f \Rightarrow \tilde{f} - B}}}$$

Global str. of gauge group

$$SU(N) \text{ YM } (a) \xrightarrow[\mathbb{Z}_N^{[1]} \text{ gauging}]{\text{dynamically}} \frac{SU(N)}{\mathbb{Z}_N} \text{ YM } (\overset{\substack{U(N) \text{ 1-form} \\ \swarrow}}{\tilde{a}}, \overset{\substack{U(1) \text{ 2-form} \\ \swarrow}}{b}) \text{ w/ } Nb = tr\hat{f}$$

$(\mathbb{Z}_N^{[1]})_{ele}$ $\downarrow$ acts on

$W(C) = tr[P e^{i\oint_C a}]$

$\underline{H(C,S)}$: 't Hooft op. is not genuine line, but it is a bdry of surface op.

$\uparrow$

This op. lives on bdry of the op. is a $(\mathbb{Z}_N^{[1]})_{ele}$ generator

acts $\circlearrowleft$ $(\mathbb{Z}_N^{[1]})_{mag}$.

$H(C)$: 't Hooft op.

$\underline{W(C,S)}$: Wilson loop lives on the bdry of surface op.

$\underline{e^{-i\int_S b}}$

$\uparrow$ $\uparrow$

Closed surface op. of $e^{i\int_{M_2} b}$ is generator of $(\mathbb{Z}_N^{[1]})_{mag}$.

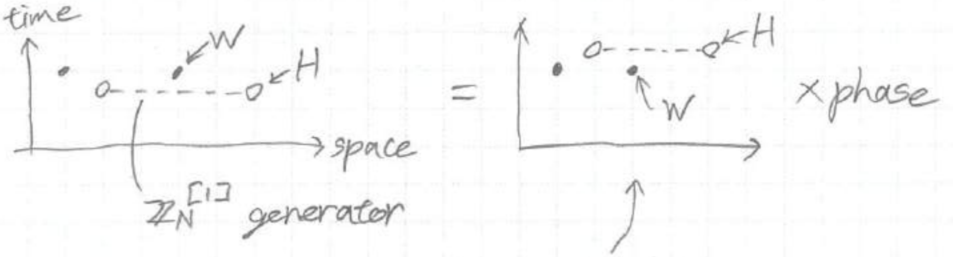
This wild surface dependence explains the Wilson-'t Hooft commutation relation between $W(C)$ and $H(C,S)$

In op. formalism, consider operators (cf. 't Hooft '78)

$$\hat{W}(C), \ \hat{H}(C')$$

then, the equal time commutation relation is

$$\hat{W}(C)\hat{H}(C') = \hat{H}(C')\hat{W}(C) \times e^{\frac{2\pi i}{N}\mathrm{link}(C,C')}$$



If we believe that both W and H are genuinely lines, this commutation relation contradict with locality.

Explicit form of the dynamical gauging:

$$Z_{\mathcal{T}}[B]$$

$$\downarrow \text{ dynamical gauging}$$

$$Z_{\mathcal{T}/\mathbb{Z}_N^{[1]}}[\underset{\uparrow}{\tilde{B}}] \longleftarrow \text{b.g. gauge field of the dual } \mathbb{Z}_N^{[1]} \text{ sym.}$$

$$= \underbrace{\int \mathcal{D}b}_{\uparrow} \, e^{\frac{iN}{2\pi}\int \tilde{B}\wedge b} \, Z_{\mathcal{T}}[b]$$

I'll often omit the full expression below.

In the previous lectures,

$$Z_{\mathcal{T}/\mathbb{Z}_N} = \int \mathcal{D}b\,\mathcal{D}c\,\delta(Nb-dc)\,Z_{\mathcal{T}}[(b,c)]$$

Gauging of $\mathbb{Z}_N$ is reversible operation

$$\mathcal{T} \text{ w/ } \mathbb{Z}_N^{[1]} \xrightarrow{\mathbb{Z}_N^{[1]} \text{ gauging}} \mathcal{T}/\mathbb{Z}_N^{[1]} \text{ w/ dual } \mathbb{Z}_N^{[1]} \big(= \overset{\overset{(d-p-2)\text{-form}}{\downarrow}}{\widetilde{\mathbb{Z}}_N^{[1]}}\big) \xrightarrow{\widetilde{\mathbb{Z}}_N^{[1]} \text{ gauging}} (\mathcal{T}/\mathbb{Z}_N^{[1]})/\widetilde{\mathbb{Z}}$$

$$Z_{\mathcal{T}}[B] \qquad Z_{\mathcal{T}/\mathbb{Z}_N^{[1]}} = \int \mathcal{D}b\, e^{\frac{iN}{2\pi}\int \tilde{B}\wedge b} Z_{\mathcal{T}}[b] \qquad \simeq \mathcal{T} \text{ w/ } \mathbb{Z}_N$$

(up to charge conj.

$$Z_{(\mathcal{T}/\mathbb{Z}_N^{[1]})/\widetilde{\mathbb{Z}}_N^{[1]}}[B] = \int \mathcal{D}\tilde{b}\, e^{\frac{iN}{2\pi}\int B\wedge \tilde{b}}\, Z_{\mathcal{T}/\mathbb{Z}_N^{[1]}}[\tilde{b}]$$

$$= \int \mathcal{D}b\,\mathcal{D}\tilde{b}\, \underbrace{e^{\frac{iN}{2\pi}\int B\wedge \tilde{b}}\, e^{\frac{iN}{2\pi}\int \tilde{b}\wedge b}}_{\overset{\shortparallel}{e^{\frac{iN}{2\pi}\int \tilde{b}\wedge(B+b)}} = \delta(B+b)}\, Z_{\mathcal{T}}[b]$$

$$= Z_{\mathcal{T}}[\underset{\uparrow \text{charge conj.}}{-}B]$$

Mixed 't Hooft anomaly b/w $\mathbb{Z}_N^{[1]}$ and θ-periodicity $\theta \sim \theta + 2\pi$ and Dashen phenomenon

(cf. TTT)

YM theory $\mathcal{L} = \frac{1}{g^2}\mathrm{tr}[f\wedge * f] + \underbrace{\frac{i\theta}{8\pi^2}\mathrm{tr}[f\wedge f]}_{\text{"almost" total derivative}}$

$\Rightarrow$ θ plays no role in perturbation theory.

Still, θ has a nontrivial effect on local perturbative dynamics (cf. 't Hooft ; Witten in the 70's)

In large-N limit, the ground state energy becomes as

$$E_{G.S.}(\theta) = \mathcal{O}(N^2) + \frac{\chi_{YM}}{2}\theta^2 + \mathcal{O}\left(\frac{\theta^4}{N^2}\right)$$

χ_{YM}: YM topological susceptibility $\sim \Lambda^4$

However, we denote that

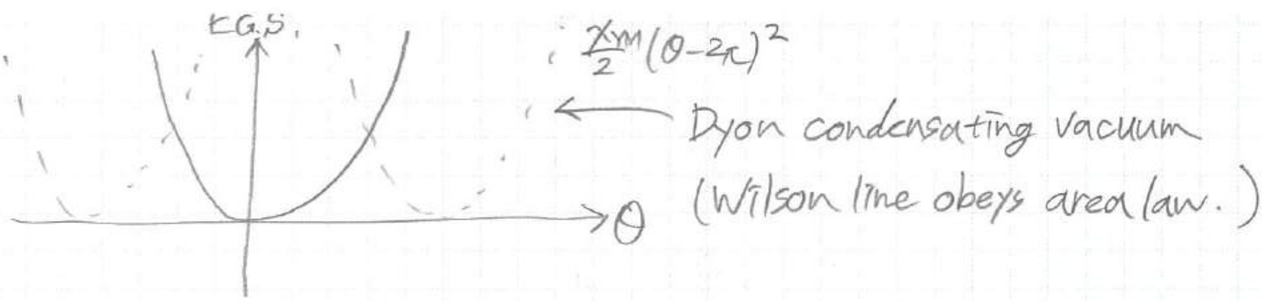
$$e^{-V\cdot E_{G.S.}(\theta)} = Z(\theta) = \int \mathcal{D}a\, e^{-\frac{1}{g^2}\int \mathrm{tr}(f\wedge * f) + \underbrace{\frac{i\theta}{8\pi^2}\int \mathrm{tr}(f\wedge f)}_{\uparrow \text{ Winding \# of } SU(N) \text{ gauge fields}}}$$

$\Rightarrow Z(\theta+2\pi) = Z(\theta)$

i.e. $\underbrace{E_{G.S.}(\theta+2\pi) = E_{G.S.}(\theta)}_{\uparrow}$

This requirement seems to be inconsistent with the large-N counting

$\Rightarrow$ multi-branch structure of YM vacua.



Maxwell action in Mink.

$$\mathcal{L} = \int F_{\mu\nu}F_{\mu\nu} + \theta\int F_{\mu\nu}\tilde{F}_{\mu\nu}$$

$$= \int(\dot{A}^2 - B^2) + \underline{\theta\int \dot{A}\cdot B}$$

$$\text{canonical conj. of } A \;:\; \pi = \frac{\delta\mathcal{L}}{\delta\dot{A}} = \overset{\mathbb{E}}{\overset{\shortparallel}{\dot{A}}} + \underline{\theta B}$$

at finite θ, $\mathbb{E}$ is different from $\mathbb{E} = \dot{A}$

We now give a "field theoretic proof" to see confining vacua at $\theta = 0$ and $\theta = 2\pi$ are disitnct as quantum phase of matter.

To this purpose, let's discuss the possible counter terms that consist only of B.

We'd like to extract a "rigit data" from $Z[B]$, to find an index distinguishing different phases.

$$\left(\text{cf. when } \mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} 1 \text{, we find } Z[B] = \delta(B) = \int \mathcal{D}b\, e^{\frac{iN}{2\pi}\int B\wedge b}\right)$$

In our current interest, $\mathbb{Z}_N^{[1]}$ is unbroken and system is gapped

$$Z[B] = e^{(\text{local Lagrangian of } B)}$$

もしbrokenだったら各vacuaをbroken $\mathbb{Z}_N^{[1]}$ labelで貼れる
unbrokenの方が難しい

Rough idea

gapped ⇒ all correlators decay exponentially fast.

unbroken ⇒ correlators satisfy cluster decomposition properties.

$$Z[A] = \sum A_1 \cdots A_n \underbrace{\langle j_1 \cdots j_n \rangle}_{\text{In IR}, \to \delta(\cdots)\delta(\cdots)}$$

$$= e^{i\int dx\,(\text{local Lagrangian of } A)}$$

⇒ It is thus important to classify possible form of local Lagrangian of B.

Thus, we consider the top. action

$$S_{top} = \frac{Nk}{4\pi}\int B\wedge B \leftarrow \left(\begin{array}{l}\text{This is a 4d analogue of CS}\\ \text{and } k\in\mathbb{Z} \text{ corresponds to CS level}\end{array}\right)$$

Requirement of the gauge inv.

Gauge trans. $B \to B + d\Lambda$

$$S_{top}[B+d\Lambda] = \frac{Nk}{4\pi}\int \left(B^2 + 2Bd\Lambda + (d\Lambda)^2\right)$$

$$= S_{top}[B] + \underbrace{\frac{k}{2\pi}\int \underbrace{NB\wedge d\Lambda}_{\overset{\shortparallel}{dC}}}_{\swarrow} + \underbrace{\frac{Nk}{4\pi}\int (d\Lambda)^2}_{\downarrow}$$

Dirac quantization

$2\pi\times\mathbb{Z}$

U(1) index thm. (on spin mfd)

$2\pi\times\mathbb{Z}$

If $k\in\mathbb{Z}$, then $S_{top}[B]$ is gauge inv.

$k\in\mathbb{Z}$ is discrete ← This cannot be continuously changed.

If k's are different b/w two confining vacua, there has to be a phase transition.

$k \sim k+N$ (Actually, $k \in \mathbb{Z}_N$ instead of $\mathbb{Z}$)

if $k = N$,

$$S_{top} = \frac{N^2}{4\pi}\int B\wedge B \overset{NB=dC}{\overset{\downarrow}{=}} \frac{1}{4\pi}\int (dC)^2 \overset{U(1)\text{ index thm.}}{\overset{\downarrow}{\in}} 2\pi\mathbb{Z}$$

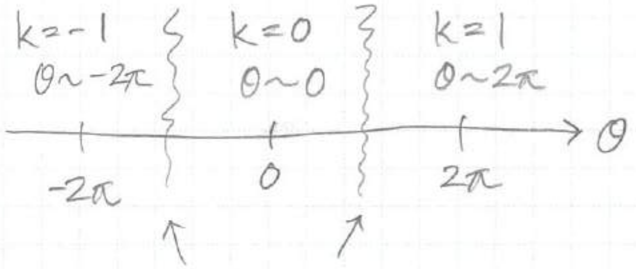
So, $e^{i\frac{N(k+N)}{4\pi}\int_{M_4} B^2} = e^{\frac{iNk}{4\pi}\int_{M_4} B^2}$ so $k \sim k+N$

Now, we derive the following relation for 4d YM (+Adj. matter)

We denote its partition func. w/ B at θ as $Z_\theta[B]$,

then $Z_{\theta+2\pi}[B] = e^{\frac{iN}{4\pi}\int B\wedge B} Z_\theta[B]$ —— $(*)$

If we assume it and YM are always confining at any θ,



There has to be phase transitions separating $\theta=0$, $\theta=\pm 2\pi$.
as long as $\mathbb{Z}_N^{[1]}$ is kept intact

2コマ目

※ These distinction by classical actions of topological gauge theories are called as SPT states.

Proof of $(*)$

We'll show that (cf. van Baal '82)

$$Q_{top}[B] = \frac{1}{8\pi^2}\int \mathrm{tr}\left[(\tilde{f}-B)\wedge(\tilde{f}-B)\right] \in \underbrace{-\frac{N}{4\pi}\int B^2}_{\frac{1}{N}\text{ quantized}} + \mathbb{Z}$$

Once it is proven then

$$Z_{\theta+2\pi}[B] = \int \mathcal{D}\tilde{a}\, e^{-\left\{\frac{1}{g^2}\int \mathrm{tr}[|\tilde{f}-B|^2] + i(\theta+2\pi)Q_{top}[B]\right\}}$$

$$= \int \mathcal{D}\tilde{a}\, e^{-\left(S_{YM} - \frac{iN}{4\pi}\int B^2\right)} = e^{\frac{iN}{4\pi}\int B^2} Z_\theta[B] \quad \leftarrow \text{This is we want to show.}$$

$$Q_{\mathrm{top}}[B] = \frac{1}{8\pi^2}\int \mathrm{tr}\left[\tilde{f}^2 - 2B\tilde{f} + B^2 \mathbb{1}_N\right]$$

$$= \frac{1}{8\pi^2}\int \mathrm{tr}[\tilde{f}^2] - \frac{1}{4\pi^2}\int B\wedge \underbrace{\mathrm{tr}\,\tilde{f}}_{\overset{\shortparallel}{NB}} + \frac{N}{8\pi^2}\int B^2$$

$$= \underbrace{\frac{1}{8\pi^2}\int \mathrm{tr}\,\tilde{f}^2}_{\xrightarrow{U(N)\text{ index thm.}}\ \mathbb{Z}} - \frac{N}{8\pi^2}\int B^2$$

$$\in -\frac{N}{8\pi^2}\int B^2 + \mathbb{Z}$$

//

The relation (*) is called as a generalized mixed 't Hooft anom. b/w $\mathbb{Z}_N^{[1]}$ and $\theta \sim \theta + 2\pi$.

- If we promote θ to a periodic axion field w/ $\theta(x) \sim \theta(x) + 2\pi \underline{\underline{N}}$

$$Z_{\mathrm{YM+axion}} = \int \mathcal{D}a\, e^{-\frac{1}{g^2}\int |f|^2 - \int |d\theta|^2 + \frac{i}{8\pi}\int \theta\, \mathrm{tr}(f\wedge f)}$$

This system has $\mathbb{Z}_N^{[1]}$ and $\underbrace{\mathbb{Z}_N^{[0]} : \theta(x) \to \theta(x) + 2\pi}_{\text{This is a global sym, not a redundancy.}}$

$Z_{\mathrm{YM+axion}}[B, A]$ has an 't Hooft anom.

$$Z_{\mathrm{YM+axion}}[B + d\Lambda^{(1)}, A + d\Lambda^{(0)}] = e^{i\frac{N^2}{2\pi\cdot 4\pi}\int \Lambda^{(0)}\wedge B^2} Z_{\mathrm{YM+axion}}[B, A]$$

This anom. can be cancelled by 5dim topological action.

$$S_{5d} = \frac{2\pi}{N}\int \frac{N}{2\pi} A\wedge \frac{N^2}{8\pi^2} B^2$$

$$= \frac{N^2}{8\pi^2}\int A\wedge B^2$$

- Another remark: If you consider $\mathcal{N}=1$ $\overset{SU(N)}{\vee}$ SYM theory ,

(SU(N) YM + massless adj. Weyl fermion)

$$\underset{B}{\mathbb{Z}_N^{[1]}} \times \underset{A}{(\mathbb{Z}_{2N}^{[0]})_{\mathrm{chiral}}} \ \leftarrow \text{often called as R-sym.}$$

There's a mixed anom.

$$S_{5d} = \frac{2\pi}{N}\int \frac{2N}{2\pi} A \wedge \frac{N^2}{8\pi^2} B^2 \quad \in \frac{2\pi}{N}\mathbb{Z}$$

$\Rightarrow$ If confining and gapped, anom. matching requires

$$\underbrace{\mathbb{Z}_{2N}^{[0]}}_{\lambda_\alpha \to e^{\frac{2\pi i}{2N}k}\lambda_\alpha} \longrightarrow \underbrace{\mathbb{Z}_2^{[0]}} \quad \Rightarrow N \text{ degenerate vacua}$$

$$\langle \lambda\lambda \rangle_k = \Lambda^3 e^{\frac{2\pi i}{N}k} \quad (k=1,\cdots,N,$$

$\uparrow$ fermion parity, so this cannot be broken.

(Lorentz-inv. vacua)

Applications of $\mathbb{Z}_N^{[1]}$ to derive the Wilson-'t Hooft classifications

't Hooft observed that

$$\hat{W}(c)\hat{H}(c') = \hat{H}(c')\hat{W}(c)\, e^{\frac{2\pi i}{N}\mathrm{Link}(c,c')}$$

$\rightsquigarrow$ (Although the following argument is incorrect.)

This relation requires the Area law for (at least) one of W or H — (**)

We now classify loop op. by $\underset{\text{mag}}{\underset{\uparrow}{\tilde{\mathbb{Z}}_N}} \times \underset{\text{ele}}{\underset{\uparrow}{\mathbb{Z}_N}} \ni (m,\ell)$

changes as $H^m W^\ell = W(m,\ell)$

$$W(m,\ell)(C)\, W(m',\ell')(C') = W(m',\ell')(C')\, W(m,\ell)(C) \cdot e^{\frac{2\pi i}{N}(m\ell' - m'\ell)\mathrm{link}(C,C}$$

If we believe (**), it suggests that at least one of $W(m,\ell)$ or $W(m',\ell')$ should obey area law unless

$m\ell' - m'\ell = 0 \mod N$ $\leftarrow$ ("mutual locality")

$\rightsquigarrow$ gapped vacua are classified by the set of deconfined lines, and they're specified by the order-N subgroup of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$:

$\uparrow$ genuine line op.

$$H_{(n,k)} = \{ \underset{\uparrow}{x}(n,0) + \underset{\nearrow}{y}\left(-k, \tfrac{N}{n}\right) \} \subset \hat{\mathbb{Z}}_N \times \mathbb{Z}_N$$

mod N integers

$$= (xn, 0) + \left(-yk, y\tfrac{N}{n}\right) = \left(xn - yk, y\tfrac{N}{n}\right)$$

$$(xn-yk)\cdot y'\frac{N}{n} - (x'n - y'k)\cdot y\frac{N}{n} = \underbrace{(xy'-x'y)}_{\in \mathbb{Z}}N + \underbrace{(-yy'+y'y)}_{=0}\frac{kN}{n} = 0 \text{ mod } N$$

In the corrected version, this argument seems not to classify the gapped vacua in it only specifies the choice of gauge grp w/ discrete θ-term.

However, it turns out that Wilson-'t Hooft classification of gapped vacua turns out to be justified for 4d QFT w/ $\mathbb{Z}_N^{[1]}$
(cf. Nguyen, Tanizaki, Unsal)

Our statement

There is a one-to-one correspondence b/w
- Wilson-'t Hooft classification

and
- SSB of $\mathbb{Z}_N^{[1]}$ sym, $\mathbb{Z}_N^{[1]} \xrightarrow{SSB} \mathbb{Z}_n^{[1]}$, stacked w/ an SPT state of the unbroken sym. (Adding level-k SPT of $\mathbb{Z}_n^{[1]}$)

Preparation and Assumption

When $\mathbb{Z}_N^{[1]} \xrightarrow{SSB} \mathbb{Z}_n^{[1]}$, the IR theory should contain a TQFT.

A minimal scenario is described by a BF theory

$$Z_{\text{eff}} = \int \mathcal{D}b\,\mathcal{D}a\; e^{\frac{i(N/n)}{2\pi}\int b\wedge da}$$

(As a generalization of coset σ-model for SSB, we should have $\underbrace{\mathbb{Z}_N/\mathbb{Z}_n}_{\simeq \mathbb{Z}_{N/n}}$ gauge theory.

In original theory, $k \in \mathbb{Z}_N^{[1]}$ acts on the Wilson loop as

$$W(C) \to e^{\frac{2\pi i}{k}} W(C)$$

So $\mathbb{Z}_N^{[1]} \xrightarrow{SSB} \mathbb{Z}_n^{[1]}$ means that

· $W(C)$ is confines (as $W \to e^{\frac{2\pi i}{n}} W$ charges $\mathbb{Z}_n^{[1]}$)

but ← This should survive in the IR limit.

· $W^n(C)$ is deconfined. (as $W^n \to W^n$ under $\mathbb{Z}_n^{[1]}$)

Therefore, we'd like to identify

$$W^n(C) \xrightarrow{\text{RG flow}} \tilde{W}(C) = e^{i\oint a} \text{ in TQFT.}$$

$$W^N \overset{\text{trivial}}{=} (W^n)^{\frac{N}{n}} \Leftrightarrow \left(\tilde{W}(C)\right)^{\frac{N}{n}} = 1 \text{ in TQFT.}$$

$$U_k(M_2) \xrightarrow{\text{RG flow}} e^{ik\int_{M_2} b}$$

We now assume the minimal scenario for EFT of $\mathbb{Z}_N^{[1]} \to \mathbb{Z}_n^{[1]}$.

Let's couple B for $\mathbb{Z}_N^{[1]}$

$$W^n(C) \to W^n(C)\, e^{in\int B}$$

$$\downarrow \text{RG flow}$$

$$\tilde{W}(C)\, e^{in\int B} = e^{i\oint_C a + in\int_S B}$$ ← This should be gauge in

So $B \to B + d\Lambda$, $a \to a - n\Lambda$

Thus, the minimal coupling procedure suggests,

$$Z_{eff}[B] = \int \mathcal{D}b\mathcal{D}a\, e^{\frac{i(N/n)}{2\pi}\int b\wedge(da+nB)} \times (\cdots)$$

potentially non-minimal part can exist.

As an extra term, we can add a local counter term of B;

$$Z_{eff}[B] = \underbrace{\int \mathcal{D}b\mathcal{D}a\, e^{\frac{i(N/n)}{2\pi}\int b\wedge(da+nB)}}_{\delta(nB)} \cdot e^{i\frac{nk}{4\pi}\int B\wedge B}$$

This is not N!

Gauge inv. of $Z_{\rm eff}[B]$

① Perform $\int \mathcal{D}b\mathcal{D}a$ integral, and then

$$Z_{\rm eff}[B] = \underbrace{\delta(nB)}_{\uparrow} \times \underbrace{e^{i\frac{nk}{4\pi}\int B\wedge B}}_{\uparrow}$$

$Z_{\rm eff}$ survives only if B is a gauge field of $\underset{\substack{\uparrow \\ \text{unbroken subgroup}}}{\mathbb{Z}_n^{[1]}} \subset \mathbb{Z}_N^{[1]}$ → Then, this is well-defined counter term.

② Equivalentlly, one can check that the following transformatic

$$\begin{cases} B \to B + d\Lambda \\ a \to a - n\Lambda \\ b \to b - \frac{k}{N/n}\, d\Lambda \end{cases} \quad \text{gives} \quad S \to S + 2\pi\mathbb{Z}$$

Claim (Reformulation of Wilson-'t Hooft classification)

For the vacua described by

$$Z_{\rm eff}[B] = \underbrace{\delta(nB)}_{\substack{\uparrow \\ \mathbb{Z}_N^{[1]} \xrightarrow{SSB} \mathbb{Z}_n^{[1]}}} \underbrace{e^{i\frac{nk}{4\pi}\int B\wedge B}}_{\text{Stacking of level-}k\text{ SPT}}$$

the deconfined lines are generated by

$$W^n(C)\,, \quad \underline{H^{\frac{N}{n}}(C)\, W^k(C)}$$

line op. with the perimeter law

The perimeter law of non-genuine lines detects the level of SPT state.

3コマ目

To prove this statement, we consider the following spacetime.

$$M_4 = M_3 \times S^1$$

$$\text{s.t.}\quad \text{size}(M_3) \gg \underbrace{\text{size}(S^1)}_{=L} \gg \frac{1}{\text{mass gap}}$$

We can use effective 4d TQFT descriptio

Furthermore, since size$(M_3) \gg$ size(S^1), we can effective regard our theory as 3d QFT defined on M_3.

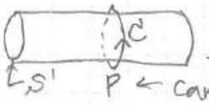
In this set up, we can decompose the b.g. gauge field B as

$$B = \underbrace{B_m}_{\substack{\uparrow \\ \text{space-space} \\ \text{component}}} + \underbrace{A \wedge \frac{d\tau}{L}}_{\substack{\uparrow \\ \text{space-time component}}}$$

We can take a gauge so that B_m, A are τ-indep.

$$\begin{cases} B_m = \mathbb{Z}_N \text{ 2-form gauge field on } M_3 = \text{b.g. of } (\mathbb{Z}_N^{[1]})_{3d} \\ A \ : \mathbb{Z}_N \text{ 1-form} \qquad 〃 \qquad = \text{b.g. of } \underbrace{(\mathbb{Z}_N^{[0]})_{3d}}_{\nearrow} \end{cases}$$

In old days, what people called the center sym.
means $(\mathbb{Z}_N^{[0]})_{3d}$ (Which acts on Polyakov loop)



Comment

In general, if we take S^1-compactified spacetime for d-dim QFT w/ $\mathbb{Z}_N^{[p]}$ sym.
Then the effective $(d-1)$-dim QFT enjoys

$$(\mathbb{Z}_N^{[p]})_{(d-1)\,\text{dim}} \times (\mathbb{Z}_N^{[p-1]})_{(d-1)\,\text{dim}}$$

$$\begin{cases} (\mathbb{Z}_N^{[1]})_{3d} \text{ acts on the spacial Wilson loop } W(C) \text{ w/ } C \subset M_3 \\ (\mathbb{Z}_N^{[0]})_{3d} \text{ acts on the temporal Wilson loop } (= \text{Polyakov loop}) \ P(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{w/ } x \subset M_3 \end{cases}$$

$$\begin{array}{l} \text{4d QFT w/ } \mathbb{Z}_N^{[1]} \\ \Big\downarrow \ \text{(large enough) } S^1\text{-compactification} \\ \text{3d QFT w/ } (\mathbb{Z}_N^{[1]})_{3d} \times (\mathbb{Z}_N^{[0]})_{3d} \\ \Big\downarrow \ (\mathbb{Z}_N^{[0]})_{3d} \text{ - gauging } \left(\begin{array}{l} \text{In our paper, we call it the temporal} \\ \text{gauging of 1-form sym.} \end{array} \right) \end{array}$$

$$\downarrow$$

$$\text{3d QFT w/ } \underset{B_m}{(\mathbb{Z}_N^{[1]})_{3d}} \times \underset{\widetilde{B_e}}{(\widetilde{\mathbb{Z}_N^{[1]}})_{3d}} \quad \xleftarrow{} \text{(magnetic) dual sym} \quad (\underset{3}{\overset{\shortparallel}{d}} - \underset{0}{\overset{\shortparallel}{p}} - 2) = 1$$

The corresponding pertition func. $(=: Z_{tH}[\widetilde{B_e}, B_m])$ is given by

$$Z_{tH}[\widetilde{B_e}, B_m] = \int \mathcal{D}a\, e^{i\frac{N}{2\pi}\int_{M_3} \widetilde{B_e}\wedge a}\, Z[\underbrace{B_m + a\wedge \tfrac{d\tau}{L}}_{B \text{ (in 4d)}}] \quad -(\bigstar$$

Comment As a general property of $Z_{tH}[\widetilde{B_e}, B_m]$, we can prove

$$Z_{tH}[\widetilde{B_e}, B_m] \geq 0 \quad \text{(i.e. real and positive)}$$

This is an outcome of reflection positivity (i.e. unitarity)

Let's compute $Z_{tH}[\widetilde{B_e}, B_m]$ for

$$Z_{\text{eff}}[B_{4d}] = \delta(nB_{4d})\, e^{i\frac{nk}{4\pi}\int B_{4d}^2}$$

$$\downarrow$$

$$Z_{\text{eff}}[B_m + A\wedge \tfrac{d\tau}{L}] = \delta(nB_m)\cdot\delta(nA)\, e^{i\frac{nk}{4\pi}\times 2\int_{M_3\times S^1} B_m\wedge A\wedge \frac{d\tau}{L}}$$

$$= \delta(nB_m)\delta(nA)\, e^{i\frac{nk}{2\pi}\int_{M_3} B_m\wedge A}$$

$$\Rightarrow Z_{tH}[\widetilde{B_e}, B_m] = \int \mathcal{D}a\, e^{i\frac{N}{2\pi}\int \widetilde{B_e}\, a}\, \delta(nB_m)\delta(na)\, e^{\frac{ink}{2\pi}\int B_m a}$$

$$= \delta(nB_m)\int \mathcal{D}a\, \delta(na)\, e^{i\frac{n}{2\pi}\int a\wedge(kB_m + \frac{N}{n}\widetilde{B_e})}$$

$$= \delta(nB_m)\,\delta(kB_m + \tfrac{N}{n}\widetilde{B_e})$$

In 3d QFT (★), we find that

$$\widetilde{\mathbb{Z}_N^{[1]}} \times \mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} \overset{\text{order-}N\text{ subgroup}}{H_{(n,k)}} = \{x(n,0) + y(-k, \tfrac{N}{n})\}$$

Comment

Under our assumption, we have shown that

$$\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \xrightarrow{SSB} \text{Order-}N \text{ subgroup}$$

This is an outcome of 4d Lorentz inv.

For example, if $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ is completely broken,

then $$\underbrace{(\mathbb{Z}_N^{[1]})_{3d}}_{\text{space part is } \underline{\text{confined}}} \times \underbrace{(\mathbb{Z}_N^{[0]})_{3d}}_{\text{temporal part is } \underline{\text{decontined}}} \xrightarrow{SSB} (\mathbb{Z}_N^{[0]})_{3d}$$

↑ This difference contradict w/ the Lorentz inv. ↑

Comment

All of these possibilities in the Wilson-'t Hooft classifications are known to be realized as gapped vacua of 4d SU(N) gauge theories.

As an explicit realization, they can be gapped vacua of 4d $\mathcal{N}=1^*$ SYM theory.

↑ Soft-breaking from $\mathcal{N}=4$ SYM to $\mathcal{N}=1$

(cf. Donagi, Witten '95)

Remark

Here, we assumed the minimal scenario for TQFT associated w/ SSB $\mathbb{Z}_N^{[1]} \longrightarrow \mathbb{Z}_n^{[1]}$.

By considering more general 4d TQFTs for this SSB, we may be able to refine the Wilson-'t Hooft classification.

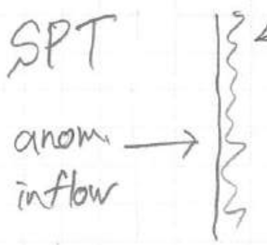
4d TQFT is much simpler than 3d TQFT.

↑ There are actually several papers claiming that 4d TQFTs are completely classified. ↖ Mathematical one by Theo Jonson-Freyed '19

$$Z_{\text{eff}}(\underset{\uparrow}{n},\underset{\uparrow}{k}) = \delta(nB)\, e^{i\frac{nk}{4\pi}\int B^2}$$

Specify unbroken sym. : This can be easily determined by area v.s. perimeter for W^{ℓ}

How can we determine the SPT label k?



← bdry theory must have non-trivial long-range excitation.
(Typical e.g. for this case is $SU(N)_k$ CS theory)

We don't need to create bulk, but determine perimeter law from non-genuine dyonic lines $H^{\frac{N}{n}} W^{\frac{k}{n}}$.
(This is reminiscent of Kennedy-Tasaki.)

Application of knowledge of 1-form sym. to QFTs w/o 1-form sym.

Let's work on an explicit e.g. $\begin{cases} \text{2d } \mathbb{CP}^{N-1} \text{ model} \\ \text{2d } N\text{-flavor Schwinger mode} \end{cases}$

$$\mathbb{CP}^{N-1} = \underbrace{S^{2N-1}}_{}/S^1$$

← We interpret it as a unit sphere in $\mathbb{C}^N \ni (z_1, \cdots, z_N) = \vec{z}$

Quatient by $S^1 \simeq U(1)$ implies the gauge identification
$\vec{z} \sim e^{i\alpha}\vec{z}$

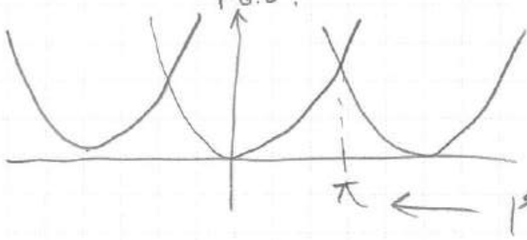
$$S = \frac{1}{g^2}\int |(d+ia)\vec{z}|^2 + \frac{i\theta}{2\pi}\int da$$

↑ auxiliary $U(1)$ gauge field. $(a = i\vec{z}^{\dagger} d\vec{z})$

Comment (in 2d)

This theory is asymptotically free

- In large-N, we can prove the mass gap.
- $N=2$ is an effective theory of $(1+1)$ d antiferro quantum Heisenberg model.
  (in $S\to\infty$ limit and $\theta = 2\pi \overset{\text{spin}}{S}$)



$\pi \leftarrow$ 1st order phase transition for $N \geq 3$
2nd order described by $SU(2)_1$ WZW for $N=2$

Let's prove this.

We first need to specify the global sym.
The Lagrangian is invariant under $\vec{z} \to \overset{SU(N)}{\overset{\in}{U}} \vec{z}$
However, when $U = e^{\frac{2\pi i}{N}} \mathbb{1}_N \in SU(N)$, then
$\vec{z} \to e^{\frac{2\pi i}{N}} \vec{z}$: This is a part of the $U(1)$ gauge trans.

$\Rightarrow$ The global sym. that faithfully acts on gauge-inv. op. is given by $SU(N)/\mathbb{Z}_N$

We want to introduce the b.g. gauge field for $SU(N)/\mathbb{Z}_N$.

$$(\hat{A}, B) : \begin{cases} \hat{A} & : U(N) \text{ gauge field} \\ B & : U(1) \text{ 2-form gauge field} \end{cases}$$

These higher-form gauge field appears w/o higher-form sym if the 0-form sym. group has a non-trivi quotients.

with the constraint $NB = d(\mathrm{tr}(\hat{A}))$

Gauge trans. $B \to B + d\Lambda$
$\hat{A} \to \hat{A} + \Lambda \mathbb{1}_N$
$a \to a - \Lambda \leftarrow$ This is required for the 1-form gauge invariance of the covariant derivative

$S_\theta[(\vec{z}, a), (\hat{A}, B)]$

$$= \frac{1}{g^2}\int |(d+ia+i\hat{A})\vec{z}|^2 + i\frac{\theta}{2\pi}\int(da+B)$$

We find the partition func. w/ $(\hat{A}, B)$ is given by

$$Z_\theta[(A,B)] = \int \mathcal{D}a\,\mathcal{D}\vec{z}^\dagger\mathcal{D}\vec{z}\, e^{-S_\theta[(\vec{z},a),(\hat{A},B)]}$$

$$\Downarrow$$

$$Z_{\theta+2\pi}[(A,B)] = \underbrace{e^{-i\int \overset{\frac{N}{2\pi}A_1\wedge A_2}{B}}} Z_\theta[(A,B)]$$

We find an extra local counter term

This is a generalized mixed 't Hooft anom. b/w $\frac{SU(N)}{\mathbb{Z}_N}$ and $\theta \sim \theta+2\pi$